International Lecture Course Fall 2022

"When Algebra meets Analysis and Number Theory"

The Algebraic and Analytic Theory of Fuchsian Differential Equations

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Outset: This course is designed for both students and (non-expert) researchers who wish to see how the study of ordinary linear differential equations combines methods from algebra, analysis and number theory. It will be given in digital form via Zoom from October 11th, 2022, until January 24, 2023. We plan to give the course on a transparent blackboard, the so-called lightboard, which allows the audience to simultaneously follow the hand-writing and the lecturer from the front.

The prospective schedule is Tuesdays, from 5 pm till 6.30. The link to the online transmission will be distributed in due time. If you are potentially interested, please send an e-mail to

herwig.hauser@univie.ac.at

or go to the course website

www.xxyyzz.cc

to register and in order to receive updated information (you may unsubscribe at any moment). We will provide certificates of attendance and for exams. Several of the collaborating universities have agreed to recognize these diploma for their proper curricula.

Topic: Around 1866, Lazarus Fuchs asked himself whether it is possible characterize those ordinary linear differential equations

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y = 0$$

with holomorphic coefficients a_i, defined in a neighborhood of 0 in $\mathbb{C}$, which admit a basis of solutions which are *moderate* in the sense that they are either *holomorphic* at 0 or converge to ∞ at most *polynomially* in any sector in $\mathbb{C}$ with vertex 0. The second case can only occur when 0 is a singularity of the equation, say, when $a_n(0) = 0$.

This was the starting point of an exciting and multi-faceted story, opening up a new field which one calls nowadays *Differential Algebra*. It was truly a vein of gold Fuchs had discovered: First, he was able to establish a purely algebraic criterion to characterize the before mentioned equations (now called *Fuchsian equations*). Further, he (and also Frobenius and Thomé) then described explicitly all solutions of these equations. Powers of logarithms appear in combinations with holomorphic functions. Using then analytic continuation of the solutions along a closed curve around 0 one constructs the associated *Monodromy Group* of the equation. Using the language of differential fields (a field like $\mathbb{C}(x)$, equipped with a derivation) one is able to interpret this group as the (differential) Galois group of the related field extension. This group gives precise information about the solutions: for instance, it is finite if and only if there is a basis of *algebraic* solutions, i.e., functions which satisfy a polynomial equation.

One can also ask when the solutions of the equation have integer coefficients (assuming that the equation is defined over $\mathbb{Z}$), thus entering number theory. The question is still not settled, with many mysterious examples and phenomena. Not to speak of the unsolved Grothendieck-Katz p-curvature conjecture, which predicts the algebraicity of the solutions by looking at the reduction of the equation modulo primes p.

These are just a few of the numerous aspects which pop up and which are fascinating to discuss and describe. In the course, we start with a systematic introduction to Fuchsian equations and then go on to some of the most striking phenomena and puzzles.

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TIMES AND DATA

Start: Tuesday, October 11, 2022

End: Tuesday, January 24, 2023

Venue: Tuesdays 5:00 - 6:30 pm on Lightboard via Zoom

Link: To be announced

Website + Registration: www.xxyyzz.cc

Contact: herwig.hauser@univie.ac.at

PROGRAM (preliminary)

October 11: Derivations, differential operators, ordinary linear differential equations, the Weyl algebra, systems of equations, differential modules.

October 18: The passage from scalar n-th order equations to first order systems, and vice versa. Cyclic vector lemma.

October 25: Singularities of differential equations, local exponents, Euler equations, normal forms, gauge transformations.

November 8: Hypergeometric equations and hypergeometric series.

November 15: The results of Fuchs, Frobenius, Thomé: the construction of a basis of solutions of Fuchsian equations.

November 22: The criterion of Fuchs for having a basis of moderate solutions.

November 29: Analytic continuation and the monodromy group. Differential Galois Theory and the Riemann-Hilbert correspondence.

December 6: Algebraic power series solutions of differential equations. The list of Schwarz. The interlacing criterion of Beukers-Heckman.

December 13: Differential equations in positive characteristic. The Grothendieck-Katz p-curvature conjecture.

January 10: Irregular singular differential equations and their solutions. The Komatsu-Malgrange index theorem.

January 17: Linear recurrences and differential equations. Integrality of the solutions.

January 24: Periods.