On Abel's Problem about Logarithmic Integrals in Positive Characteristic

Florian Fürnsinn, Herwig Hauser and Hiraku Kawanoue

January 24, 2024

Abstract

Linear differential equations with polynomial coefficients over a field K of positive characteristic p with local exponents in the prime field have a basis of solutions in the differential extension $\mathcal{R}_p = K(z_1, z_2, \ldots)(\!(x)\!)$ of $K(x)$, where $x' = 1, z_1' = 1/x$ and $z_i' = z_{i-1}'/z_{i-1}$. For differential equations of order 1 it is shown that there exists a solution y whose projections $y|_{z_{i+1}=z_{i+2}=\cdots=0}$ are algebraic over the field of rational functions $K(x, z_1, \ldots, z_i)$ for all i. This can be seen as a characteristic p analogue of Abel's problem about the algebraicity of logarithmic integrals. Further, the existence of infinite product representations of these solutions is shown. As a main tool p^i-curvatures are introduced, generalizing the notion of the p-curvature.

1 Introduction

Niels Abel asked for criteria when a differential equation of the form

$$\frac{y'}{y} = a \tag{1}$$

has, for a a complex polynomial or a rational (respectively, algebraic) function, an algebraic solution y (see Boulanger [Bou97, p. 93]). A necessary condition is that a has only simple poles, as is the case for $\frac{y'}{y}$, for any holomorphic or meromorphic y. For instance, if b is a rational function and $k \in \mathbb{Z} \setminus \{0\}$, then $a := \frac{1}{k}\frac{b'}{b}$ yields the algebraic solution $y = \sqrt[k]{b}$. Algorithmically, the problem has been solved by Risch [Ris70]. In the present note, we address a similar problem for first order differential equations defined over a field K of positive characteristic p. A first distinction is the fact that equations like (1) need not have formal power series solutions, or, more generally, solutions of the form $x^\rho f$ for some power series $f \in K[\![x]\!]$ and some $\rho \in \overline{K}$. For instance, the exponential function $\exp \in \mathbb{Q}[\![x]\!]$, solution of $y' = y$, cannot

---

MSC2020: 34A05, 47E05. The first and second authors were supported by the Austrian Science Fund FWF P-34765, which also enabled the organisation of a workshop in Lisbon in February 2023, where the three authors started their discussion. The third author was supported by JSPS Grant-in-Aid for Scientific Research (C), No. 20K03546. We thank A. Bostan for posing Problem 1.1 for the exponential differential equation and X. Caruso, D. van Straten and S. Yurkevich for fruitful discussions and for very valuable insights. Further, we thank S. Schneider for reading a preliminary version and helping with the computations.

be reduced modulo p to obtain a solution in $\mathbb{F}_p[\![x]\!]$, for any prime p, as all prime numbers appear in the denominators of the coefficients. And, indeed, making an unknown ansatz $y = \sum_i c_i x^i$ for the solution in characteristic p, and solving for the coefficients $c_i \in K$ recursively, a contradiction occurs once i reaches p.

In [FH23] Fürnsinn and Hauser introduce the differential extension

$$\mathcal{R}_p = \mathbb{F}_p(z_1, z_2, \ldots)(\!(x)\!)$$

of the field of formal Laurent series $\mathbb{F}_p(\!(x)\!)$ by adjoining countably many variables z_i, equipped with the derivation given by

$$\partial 1 = 0, \qquad \partial x = 1,$$
$$\partial z_1 = \frac{1}{x}, \qquad \partial z_i = \frac{\partial z_{i-1}}{z_{i-1}} = \frac{1}{x \cdot z_1 \cdots z_{i-1}} \quad \text{for } i > 1.$$

The ring of constants is $\mathcal{C}_p := \mathcal{R}_p^p = \mathbb{F}_p(z_1^p, z_2^p, \ldots)(\!(x^p)\!)$. It is proven that any differential equation $Ly = 0$, for $L \in \mathbb{F}_p[\![x]\!][\partial]$ an operator of order n with regular singularity at 0 and local exponents in the prime field, has n $\mathcal{C}_p$-linearly independent solutions in $\mathcal{R}_p$, and even in $\mathbb{F}_p[z_1, z_2, \ldots][\![x]\!]$, the ring of power series in x with polynomial coefficients in the z_i.

For elements in $\mathcal{R}_p$, the notion of algebraicity is more subtle. In general, a solution y of $Ly = 0$ in $\mathcal{R}_p$ will depend on infinitely many variables z_i, so y will almost never be algebraic over $\mathbb{F}_p(x, z_1, z_2, \ldots)$. But, for a given $y \in \mathbb{F}_p[z_1, z_2, \ldots][\![x]\!]$, it is interesting to ask if at least its *projections* $y|_{z_{i+1}=z_{i+2}=\cdots=0}$ obtained by setting almost all z-variables equal to 0, are algebraic over $\mathbb{F}_p(x, z_1, z_2, \ldots, z_i)$. Said differently, can y be approximated by algebraic series, involving more and more z-variables? This suggests:

Problem 1.1. *Let $Ly = 0$ be a differential equation with regular singularity at 0 and polynomial or algebraic power series coefficients. Does there exist a $\mathcal{C}_p$-basis of solutions $y_1, \ldots, y_n \in \mathbb{F}_p[z_1, z_2, \ldots][\![x]\!]$ for which all projections $y_j|_{z_{i+1}=z_{i+2}=\cdots=0}$ are algebraic over $\mathbb{F}_p(x, z_1, \ldots, z_i)$?*

In particular, one may ask whether the *initial series* $y_j|_{z_1=z_2=\cdots=0} \in \mathbb{F}_p[\![x]\!]$ of the basis are algebraic?

For first order differential equations the answer is positive:

Theorem 1.2. *Let $y' - ay = 0$ be a linear differential equation of order 1, regular at 0, with local exponent $\rho = 0$ for some algebraic $a \in \mathbb{F}_p(\!(x)\!)$. Then there is a non-zero solution $y \in \mathbb{F}_p[z_1, z_2, \ldots][\![x]\!]$ which has algebraic projections $y|_{z_{i+1}=\cdots=0}$ for all i.*

One even has an infinite product decomposition of the solution:

Theorem 1.3. *In the preceding situation, the specified solution y can be written as a product*

$$y = \prod_{i=0}^{\infty} h_i,$$

where the factors h_i belong to $1 + x^{p^i} z_i \mathbb{F}_p[z_1, z_2, \ldots, z_i][\![x]\!]$ and are algebraic over $\mathbb{F}_p(x, z_1, \ldots, z_i)$.

Structure of the paper. In Section 2 we revise the basic setup for solving differential equations in $\mathcal{R}_p$. Then variations and consequences of Problem 1.1 will be dsicussed.

Section 3 is concerned with the exponential function in positive characteristic. Solutions of $y' = y$ are only unique up to multiplication with elements in $\mathcal{C}_p$. We will choose and construct a distinguished solution, denoted by $\exp_p$, according to the theory described in Section 2. It is then shown that for this solution $\exp_p$, the projections $\exp_p|_{z_i=z_{i+1}=\cdots=0}$ are algebraic for all i. The proof makes a small detour: One shows that another, specifically chosen solution $\widetilde{\exp}_p$ of $y' = y$, admits an infinite product decomposition with algebraic factors. And then a general result (see 2.4) implies that also $\exp_p$ must have been algebraic.

Arbitrary first order differential equations will then be addressed in Section 4, aiming at a proof of Theorem 1.2. We will generalize the concept of the p-curvature by introducing higher curvatures, called p^i*-curvature*, for any $i \geq 1$. These curvatures share many properties with the classical p-curvature, but take into account the variables z_i and yield finer information. This is then used in Section 5 to develop the proof of Theorem 1.2. In the final part, Section 6, we investigate the second order equations $y'' = \pm y$ in order to compare the characteristic p trigonometric functions with the exponential function.

2 On Fuchs' Theorem in Positive Characteristic

In this section, we first recall the main definitions and results from [FH23], reformulate them to fit our needs in this paper and make Problem 1.1 more precise.

The theory of power series solutions of linear homogeneous differential equations in characteristic p involving logarithms was initiated by Honda [Hon81]. Dwork [Dwo90] studied the case of nilpotent p-curvature and Fürnsinn and Hauser established the complete description of the solutions of arbitrary differential equations with regular singularities in [FH23]. In all three cases one has to introduce certain differential extensions of $K[\![x]\!]$.

Let us fix some notation. Let $p = \operatorname{char} K$ be a prime number and

$$L = a_n\partial^n + a_{n-1}\partial^{n-1} + \ldots + a_1\partial + a_0 \in K[\![x]\!][\partial]$$

be a differential operator with power series coefficients $a_i \in K[\![x]\!]$ over a field of characteristic p. We assume L to be regular at 0, i.e., that the quotient a_i/a_n has a pole of order at most $n-i$ in 0. Write $L = \sum_{j=0}^{n}\sum_{i=0}^{\infty} c_{ij}x^i\partial^j$ with $c_{ij} \in K$. We define the *initial form* L_0 of L as the operator

$$L_0 = \sum_{i-j=\tau} c_{ij}x^i\partial^j,$$

where τ is the minimal *shift* $i-j$ of L. We will restrict without loss of generality to differential operators with minimum shift $\tau = 0$; this can be achieved by multiplication of L with a suitable power of x. Consequently $L_0(x^k) = \chi_L(k)x^k$, where $\chi_L \in K[s]$ is the *indicial polynomial* of L; its roots ρ are called the *local exponents* of L.

In this text we will assume for simplicity that $K = \mathbb{F}_p$ and that the local exponents belong to $\mathbb{F}_p$, i.e., that the indicial polynomial splits over $\mathbb{F}_p$. For general fields and local exponents, the theory extends as described in [FH23], where the added difficulties are mostly being of technical and notational nature.

As explained in Section 1, define $\mathcal{R}_p \coloneqq \mathbb{F}_p(z_1, z_2, \dots)(\!(x)\!)$ as the field of Laurent series in x with rational functions in countably many variables z_i as coefficients equipped with the aforementioned derivation. This derivation rule resembles the differentiation of the iterated (complex) logarithm $\log(\dots\log(x)\dots)$. We will therefore call the variables z_i colloquially *logarithms*. The definition is motivated by the need to provide for any element of $\mathcal{R}_p$ a primitive under the derivation. For example, while x^{p-1} does not have a primitive in $\mathbb{F}_p[\![x]\!]$, we have $(x^p z_1)' = x^{p-1} \in \mathcal{R}_p$. As usual we will write y' instead of ∂y for $y \in \mathcal{R}_p$. The field of constants of $\mathcal{R}_p$ turns out to be $\mathcal{C}_p \coloneqq \mathbb{F}_p(z_1^p, z_2^p, \dots)(\!(x^p)\!)$ [FH23, Prop. 3.3].

Any differential operator $L \in \mathbb{F}_p[\![x]\!][\partial]$ defines a $\mathcal{C}_p$-linear map

$$L : \mathcal{R}_p \to \mathcal{R}_p, \quad y \mapsto L(y),$$

applying L to series $y \in \mathcal{R}_p$. Similarly, its initial form L_0 and its tail $T = L_0 - L$ define $\mathcal{C}_p$-linear maps. We represent the local exponents for convenience by integers between 0 and $p-1$. With this convention, it is is easy to see that the monomial

$$x^\rho z^{i^*} \quad \text{for } \rho \text{ a local exponent and } 0 \le i \le m_\rho - 1,$$

with exponents i^* defined by

$$i^* = (i, \lfloor i/p \rfloor, \lfloor i/p^2 \rfloor, \dots) \in \mathbb{N}^{(\mathbb{N})}$$

form a monomial $\mathcal{C}_p$-basis of the kernel $\operatorname{Ker} L_0$ of L_0, i.e., of the solution space of $L_0 y = 0$ in $\mathcal{R}_p$ [FH23, Prop. 3.9]. Here z^α for $\alpha \in \mathbb{N}^{(\mathbb{N})}$ denotes $z_1^{\alpha_1} \cdots z_k^{\alpha_k}$ where $k \in \mathbb{N}$ is maximal, such that $\alpha_k \neq 0$.

To formulate Fuchs' Theorem in positive characteristic, it is necessary to choose a direct complement $\mathcal{H}$ of $\operatorname{Ker} L_0$ in $\mathcal{R}_p$ as a $\mathcal{C}_p$-vector space, say, of the solution space of the Euler equation $L_0 y = 0$ associated to $Ly = 0$. There are several choices for $\mathcal{H}$, and we will discuss one particular below. The restriction $L_0|_{\mathcal{H}}$ of L_0 to $\mathcal{H}$ defines an isomorphism onto the image, which is shown to be again $\mathcal{R}_p$, using the fact that the differential field $\mathcal{R}_p$ contains sufficiently many primitives. Let $S : \mathcal{R}_p \to \mathcal{H}$ be the inverse of $L_0|_{\mathcal{H}}$, i.e., a section (or right inverse) of L_0. We get a $\mathcal{C}_p$-linear map

$$v : \mathcal{R}_p \to \mathcal{H}, \quad y \mapsto v(y) = \sum_{k=0}^{\infty} (ST)^k(y).$$

It is well defined because the composition $ST = S \circ T$ increases the order in x of a series in $\mathcal{R}_p$, thus $\sum_{k=0}^{\infty} (ST)^k(y)$ converges to a formal series.

In this setting, one has the following extension of Fuchs' theorem to the case of linear differential equations defined over a field of positive characteristic. We give here a simplified version, for the general statement, see [FH23, Thm. 3.16, Thm. 3.17]

Theorem 2.1 (Fuchs' Theorem in positive characteristic, [FH23])**.**

(i) Let $L \in K[\![x]\!][\partial]$ be a differential operator with shift 0 and local exponent $\rho \in \mathbb{F}_p$ at 0. Decompose $L = L_0 - T$ into its initial operator $L_0 \in K[x][\partial]$ and tail operator $T \in K[x][\partial]$. Let $\mathcal{H}$ be a direct complement of Ker L_0 *in $\mathcal{R}_p$, and let $S = (L_0|_{\mathcal{H}})^{-1}$ be defined as before. Then*

$$y(x) = v(x^\rho) = \sum_{k=0}^{\infty} (ST)^k (x^\rho) \in \mathcal{R}_p$$

is a solution of $Ly = 0$.

(ii) Assume that L has a regular singularity and all local exponents of L are in $\mathbb{F}_p$. The series $y_{\rho,i} := v(x^\rho z^{i^*})$ form a $\mathcal{C}_p$-basis of solutions of $Ly = 0$ in $\mathcal{R}_p$, where ρ ranges over all local exponents, m_ρ denotes the multiplicity of ρ as a local exponent, and $0 \leq i \leq m_\rho - 1$.

By abuse of notation we have written ρ for the local exponent of L in $\mathbb{F}_p$, as well as for its representative in $\{0, 1, \dots, p-1\} \subseteq \mathbb{Z}$.

The nilpotence of the p-curvature of the equation $Ly = 0$ is equivalent to having a basis $y_1, \dots, y_n \in \mathcal{R}_p$ which only depends on finitely many of the variables z_i [Hon81; Dwo90].

For $j \in \mathbb{F}_p$ and $\gamma \in \mathbb{F}_p^{(\mathbb{N})}$ we define the *section operators* $\langle \cdot \rangle_{j,\gamma} : \mathcal{R}_p \to \mathcal{R}_p$ extracting those monomials $x^k z^\alpha$ of an element $f \in \mathcal{R}_p$, such that $k \equiv j \bmod p$ and $\alpha_i \equiv \gamma_i \bmod p$ for all i. More explicitly:

$$\left\langle \sum c_{k,\alpha} x^k z^\alpha \right\rangle_{j,\gamma} := \sum_{\substack{k \in j + p\mathbb{Z} \\ \alpha \in \gamma + (p\mathbb{Z})^{(\mathbb{N})}}} c_{k,\alpha} x^k z^\alpha.$$

Note that $x^{-j} z^{-\gamma} \langle y \rangle_{j,\gamma} \in \mathcal{C}_p$.

By Theorem 2.1, any regular singular differential equation $Ly = 0$ admits a $\mathcal{C}_p$-basis of solutions, and in part (ii) of the theorem the construction of a specific basis is described in terms of an algorithm. It turns out that the resulting basis can be described intrinsically by conditions on the exponents of the involved series $y_{\rho,i}$. This works as follows.

A series $y = y_{\rho,i} \in \mathcal{R}_p$ will be called *xeric* with respect to $Ly = 0$ if there is a local exponent ρ of L and an index $0 \leq i < m_\rho$ such that

$$\langle y_{\rho,i} \rangle_{\rho,i^*} = x^\rho z^{i^*}$$

and

$$\langle y_{\rho,i} \rangle_{\sigma,j^*} = 0$$

for all pairs $(\sigma, j) \neq (\rho, i)$ of local exponents σ and indices $0 \leq j < m_\sigma$. This signifies that aside from the *initial monomial* $x^\rho z^{i^*}$ there occurs no p-th power multiple of some $x^\sigma z^{j^*}$ in the expansion $\sum c_{k,\alpha} x^k z^\alpha$ of y. This description explains the choice of the term "xeric" in the sense of "deprived of". Bases of xeric solutions of differential equations with regular singularities always exist and are then unique. In fact, it suffices to apply Theorem 2.1 in the case where the direct complement $\mathcal{H}$ of $\operatorname{Ker} L_0$ is chosen such that the power series expansion of any $y \in \mathcal{H}$ involves none of the monomials generating $\operatorname{Ker} L_0$.

For the case of order 1 differential operators with local exponent $\rho = 0$ (necessarily of multiplicity 1, hence $i = 0$ and also $i^* = 0$), the xeric solution is the unique solution y whose expansion involves no p-th power monomial except for the constant 1.

Example 2.2. We consider the series $\exp_p$, $\log_p(1-x)$, $\sin_p(x)$, and $\cos_p(x)$ in the characteristic p setting, that is, the xeric solutions of

$$y' = y, \quad xy'' - y' - x^2 y'' = 0, \quad \text{and} \quad y'' = -y,$$

respectively. Taking $p = 3$, one obtains

$$\begin{aligned}
\exp_3 &= 1 + x + 2x^2 + 2x^3 z_1 + (2z_1 + 1)x^4 + x^5 z_1 + 2x^6 z_1^2 + (2z_1^2 + 2z_1 + 1)x^7 + \\
&\quad (z_1^2 + 2)x^8 + (z_1^3 z_2 + 2z_1)x^9 + (z_1^3 z_2 + 2z_1^2 + z_1 + 2)x^{10} + \ldots \\
\log_3(1-x) &= x + 2x^2 + x^3 z_1 \\
\sin_3 &= x + 2z_1 x^3 + z_1 x^5 + (2z_1^2 + 2z_1)x^7 + z_1^3 z_2 x^9 + (2z_1^3 z_2 + z_1)x^{11} + \ldots \\
\cos_3 &= 1 + 2x^2 + 2x^4 z_1 + (2z_1^2 + z_1)x^6 + (z_1^2 + 2z_1 + 2)x^8 + \ldots
\end{aligned}$$

The coefficients c_i of Laurent series $\sum_{i=i_0}^\infty c_i(z) x^i$ in $\mathcal{R}_p$ are rational functions in the variables $z_1, z_2, \ldots$ As such, each of them depends only on finitely many variables (by definition of polynomials and rational functions in infinitely many variables), but this number may increase with the exponent k of x and actually go to ∞. We will be interested in series as above which involve only *finitely many* z-variables, say, in the subrings

$$\mathcal{R}_p^{(k)} := \mathbb{F}_p(z_1, \ldots, z_k)(\!(x)\!)$$

of $\mathbb{F}_p(z)(\!(x)\!)$, for $k \geq 0$. Restricting to polynomial coefficients in z and setting $z_{k+1} = z_{k+2} = \ldots = 0$ we get projection maps

$$\begin{aligned}
\pi_k : \mathbb{F}_p[z](\!(x)\!) &\to \mathbb{F}_p[z_1, \ldots, z_k](\!(x)\!), \\
y(x, z) &\mapsto y(x, z)|_{z_{k+1} = z_{k+2} = \ldots = 0} = y(x, z_1, \ldots, z_k, 0, \ldots,).
\end{aligned}$$

These extend naturally to projection maps

$$\pi_k : \mathbb{F}_p(z)(\!(x)\!) \to \mathbb{F}_p(z_1, \ldots, z_k)(\!(x)\!),$$

denoted by the same letter π_k. In particular, for $k = 0$ and $y \in \mathbb{F}_p[z](\!(x)\!)$, we get

$$\pi_0(y(x, z)) = y(x, 0) = \sum_{i=i_0}^\infty c_i(0) x^i \in \mathbb{F}_p(\!(x)\!),$$

called the *initial series* y_0. To cover also the case where $y \in \mathbb{F}_p(z)((x))$ has rational function coefficients, we declare $c_i(0) := c_{i0}$ to denote the constant summand in the Laurent expansion $c_i(z) = \sum_{\alpha \in \mathbb{Z}^{(\mathbb{N})}} c_{i\alpha} z^\alpha$ of $c_i(z)$. For arbitrary $k \geq 0$, we call $\pi_k(y)$ the *k-th projection* of y.

Remark 2.3. It turns out that in Fuchs' Theorem 2.1 one may specify more accurately the subspace of $\mathbb{F}_p(z)((x))$ in which the solutions live. To this end, introduce, for every $k \geq 1$, the monomials

$$w_k := z_1^{p^{k-1}} z_2^{p^{k-2}} \cdots z_{k-1}^p z_k^1.$$

Thus, $w_1 = z_1$, $w_2 = z_1^p z_2$, $w_3 = z_1^{p^2} z_2^p z_3$, and so on. It was shown in [FH23] that a basis of solutions of $Ly = 0$ already exists in $x^\rho \mathbb{F}_p[w_1, w_2, w_3, \ldots][\![x]\!]$. Further, one might restrict the space even further, by bounding the degree of the variables w_i in each monomial in terms of the degree of x.

For differential equations of order 1 with local exponent $\rho = 0$ this corresponds to the following construction: Define the ring

$$\mathcal{S}_p = \mathbb{F}_p\{x, x^p w_1, x^{p^2} w_2, \ldots\}$$

as the closure of $\mathbb{F}_p[x, x^p w_1, x^{p^2} w_2, \ldots]$ in $\mathbb{F}_p(z)((x))$ with respect to the x-adic topology. For example, infinite sums of the form

$$\sum_{k=0}^{\infty} b_k x^{p^k} w_k = \sum_{k=0}^{\infty} b_k x^{p^k} z_1^{p^{k-1}} z_2^{p^{k-2}} \cdots z_{k-1}^p z_k^1$$

belong to $\mathcal{S}_p$, for any $b_k \in \mathbb{F}_p$, since the sum converges to an element of $\mathbb{F}_p(z)((x))$ and this will be illustrated again in the later sections. As before, we may restrict to finitely many z-variables, and then set

$$\mathcal{S}_p^{(k)} = \mathbb{F}_p\{x, x^p w_1, x^{p^2} w_2, \ldots, x^{p^k} w_k\} = \mathcal{S}_p \cap \mathcal{R}_p^{(k)}.$$

In the following paragraphs, we want to make Question 1.1 more precise. Recall that the solutions of $Ly = 0$ form an n-dimensional $\mathcal{C}_p$-vector space. In particular, if y is a solution, such that its initial series is algebraic, multiplying y by a transcendent power series in x^p gives another solution of $Ly = 0$, whose initial series cannot be algebraic. However, it turns out that if a given differential equation $Ly = 0$ has a basis of power series solutions with algebraic projections, the same holds true for its xeric basis.

Proposition 2.4. *Let $L \in \mathbb{F}_p[\![x]\!][\partial]$ be a differential operator of order n and assume there is a basis $\widetilde{y}_1, \ldots, \widetilde{y}_n \in \mathbb{F}_p[z_1, z_2, \ldots][\![x]\!]$ of solutions of $Ly = 0$ with algebraic projections for all $k \in \mathbb{N}$, i.e., $\pi_k(\widetilde{y}_i)$ is algebraic over $\mathbb{F}_p(x, z_1, \ldots, z_k)$ for all k. Then, the xeric basis $y_1, \ldots, y_n$ has algebraic projections as well.*

For the proof we need the following lemma, generalizing a well known fact about sections of algebraic power series in characteristic p:

Lemma 2.5. *Let $f \in \mathbb{F}_p(z_1, z_2, \dots, z_k)((x))$. Then f is algebraic over $\mathbb{F}_p(x, z_1, z_2, \dots, z_k)$ if and only if $\langle f \rangle_{j,\alpha}$ is algebraic for all $j \in \mathbb{F}_p, \alpha \in \mathbb{F}_p^k$.*

Proof. Since f is a sum over all its sections, the condition is obviously sufficient.

To see that it is necessary, we use backwards induction over

$$e(j, \alpha) \coloneqq \overline{j} + \overline{\alpha_1}p + \overline{\alpha_2}p^2 + \dots,$$

where $\bar{\cdot} : \mathbb{Z} \to \{0, 1, \dots, p-1\}$ denotes the reduction modulo p. Note that since y only contains monomials in the variables $x, z_1, \dots, z_k$, we have $e(j, \alpha) \le p^{k+1} - 1$ for each monomial $x^j z^\alpha$ in y. The quantity $e(j, \alpha)$ is the unique integer, such that $(x^j z^\alpha)^{(e(j,\alpha))} = 0$, but $(x^j z^\alpha)^{(e(j,\alpha)-1)} \neq 0$. Moreover, $\langle f \rangle_{-1,(-1,\dots,-1)}$ is the sum of monomials $x^j z^\alpha$ in f with $e(j, \alpha) = p^{k+1} - 1$. We obtain,

$$f^{(p^{k+1}-1)} = a(e(p-1, \dots, p-1))x^{-(p^{k+1}-1)} z_1^{-(p^k-1)} \dots z_k^{-(p-1)} \langle f \rangle_{-1,(-1,\dots,-1)},$$

where $a(e(p-1, \dots, p-1))$ is a non-zero constant. This equality essentially follows from the repeated application of the product rule, always choosing to take the derivative with respect to x, except if the exponent of x is divisible by p, compare to [FH23, Lem. 3.4].

Using the fact that the any derivative of an algebraic series is algebraic, we get algebraicity of $\langle f \rangle_{-1,(-1,\dots,-1)}$. To get algebraicity of other sections, consider now $\widetilde{f} = f - \langle f \rangle_{-1,(-1,\dots,-1)}$. Then any monomial $x^j z^\alpha$ in $\widetilde{f}$ satisfies $e(j, \alpha) \le p^{k+1} - 2$ and we get analogously

$$\widetilde{f}^{(p^{k+1}-2)} = a(e(p-2, \dots, p-1))x^{-(p^{k+1}-2)} z_1^{-(p^k-1)} \dots z_k^{-(p-1)} \langle \widetilde{f} \rangle_{-2,(-1,\dots,-1)}.$$

Algebraicity of the section $\langle \widetilde{f} \rangle_{-2,(-1,\dots,-1)} = \langle f \rangle_{-2,(-1,\dots,-1)}$ follows as above. Repeating this construction of removing sections we already know to be algebraic, we prove algebraicity of all sections. $\square$

Proof of Proposition 2.4. Assume that the local exponents of L are $\rho_1, \rho_2, \dots, \rho_\ell \in \mathbb{F}_p$ of multiplicities $m_1, \dots, m_\ell$ respectively. Again, by abuse of notation, we write ρ for both the local exponent of L in $\mathbb{F}_p$ and its representative in $\{0, 1, \dots, p-1\} \subseteq \mathbb{Z}$. Without loss of generality assume that $\rho_1 < \rho_2 < \dots < \rho_\ell$ in $\mathbb{Z}$. Let $Y = (y_1, \dots, y_n)^\top$ and $\widetilde{Y} = (\widetilde{y}_1, \dots, \widetilde{y}_n)^\top$. Then there is $D \in GL_n(\mathcal{C}_p)$, such that $D\widetilde{Y} = Y$.

Truncating at a suitable order of x, there is a matrix of rational functions $\widehat{D} \in \mathrm{GL}_n(\mathbb{F}_p(x^p, z_1^p, \dots))$, such that $\widehat{D}\widetilde{Y} = \widehat{Y}$, with

$$\widehat{Y} = (x^{\rho_1}, \dots, x^{\rho_1} z^{(m_{\rho_1}-1)^*}, \dots, x^{\rho_\ell}, \dots, x^{\rho_\ell} z^{(m_{\rho_\ell}-1)^*})^\top + \dots$$

where all other terms of each row have higher order in x. In other words we choose an approximation of D of high enough degree, such that the terms of least order in x of $D\widetilde{Y}$ agree with the terms of least order in x of Y. Then there is m, such that $\widehat{D} \in \mathrm{GL}_n(\mathbb{F}_p(x^p, z_1^p, \dots, z_m^p))$, as all its entries are rational functions and thus depend only on finitely many variables z_i.

Now, $\widehat{Y}$ is again a basis of solutions so we get $C \in \mathrm{GL}_n(\mathcal{C}_p)$ such that $\widehat{Y} = CY$. By definition of the xeric solutions, we have

$$\langle Y \rangle_{\rho_s, j^*} = x^{\rho_s} z^{j^*} e_{m_1+\ldots+m_s+j}$$

for $1 \le s \le k, j < m_s$, where e_i denotes the i-th unit vector. As $C \in \mathcal{C}_p = \mathbb{F}_p(z_1^p, z_2^p, \ldots)(\!(x^p)\!)$,

$$\langle \widehat{Y} \rangle_{\rho_s, j^*} = \langle CY \rangle_{\rho_s, j^*} = C x^{\rho_s} z^{j^*} e_{m_1+\ldots+m_s+j} = x^{\rho_s} z^{j^*} C_{m_1+\ldots+m_s+j},$$

where C_i denotes the i-the column of C. But then

$$C_{m_1+\ldots+m_s+j} = x^{-\rho_s} z^{-j^*} \langle \widehat{Y} \rangle_{\rho_s, j^*} \in (\mathbb{F}_p(z_1, \ldots, z_m)[z_{m+1}, \ldots][\![x]\!])^n,$$

since, by construction the terms, in $\langle \widehat{Y} \rangle_{\rho_s, j^*}$ of lowest order in x are either 0, $x^{\rho_s} z^{j^*}$ or have order higher than p in x.

Moreover, comparing orders in x in the equation $\widehat{Y} = CY$, and using $\mathcal{C}_p$-linear independence of the terms of least order in x of Y, we obtain the following: The matrix C has entries in $1 + x(\mathbb{F}_p(z_1, \ldots, z_m)[z_{m+1}, \ldots][\![x]\!])$ on the main diagonal, the entries below the diagonal have order in x strictly greater than 0 and the entry above the main diagonal have non-negative order. Consequently,

$$\det C \in 1 + x(\mathbb{F}_p(z_1, \ldots, z_m)[z_{m+1}, \ldots][\![x]\!]) \subseteq (\mathbb{F}_p(z_1, \ldots, z_m)[z_{m+1}, \ldots][\![x]\!])^\times$$

and $C \in \mathrm{GL}_n(\mathbb{F}_p(z_1, \ldots, z_m)[z_{m+1}, \ldots][\![x]\!])$.

Recall that $\widehat{D}\widetilde{Y} = \widehat{Y}$ with $\widehat{D} \in \mathrm{GL}_n(\mathbb{F}_p(x^p, z_1^p, \ldots, z_m^p))$. So we have

$$\pi_i(\widehat{Y}) = \widehat{D} \pi_i(\widetilde{Y})$$

for $i \ge m+1$ and therefore each entry of $\pi_i(\widehat{Y})$ is algebraic, as each entry of $\pi_i(\widetilde{Y})$ is by assumption and the entries of $\widehat{D}$ are rational functions. Moreover, $Y = C^{-1}\widehat{Y}$ and using that $C^{-1} \in \mathrm{GL}_n(\mathbb{F}_p(z_1, \ldots, z_m)[z_{m+1}, \ldots][\![x]\!])$, we obtain

$$\pi_i(Y) = \pi_i(C^{-1}) \pi_i(\widehat{Y}).$$

Recall that $C_{m_1+\ldots+m_s+j} = x^{-\rho_s} z^{-j^*} \langle \widehat{Y} \rangle_{\rho_s, j^*}$. Consequently, the entries of $\pi_i(C)$ are monomial multiples of sections of $\pi_i(\widehat{Y})$, which are algebraic by Lemma 2.5. With $\pi_i(C^{-1}) = \pi_i(C)^{-1}$ we can conclude that $\pi_i(Y)$ is algebraic for $i \ge m+1$. Finally, for $i \le m$, we can apply π_i to the minimal polynomial of $\pi_{m+1}(Y) \in \mathbb{F}_p[z_1, \ldots, z_m][\![x]\!]$ and therefore $\pi_i(Y)$ is algebraic as well for these values of i.

□

Thus, we have essentially reduced Problem 1.1 to the following:

Problem 2.6. *For which differential operators $L \in \mathbb{F}_p[x][\partial]$ has the basis of xeric solutions with algebraic projections?*

We conclude this section with an assertion about the algebraicity of projections:

Lemma 2.7. *Let $f \in \mathbb{F}_p[z_1, z_2, \ldots, z_k][\![x]\!]$ be algebraic over $\mathbb{F}_p(x, z_1, z_2, \ldots, z_k)$. Write $f = \sum f_\alpha z^\alpha$ for $f_\alpha \in \mathbb{F}_p[\![x]\!]$. Then f_α is algebraic over $\mathbb{F}_p(x)$ for all α.*

Proof. We use induction on the number of variables k. The case $k = 0$ is trivial, so assume we have proven the statement for $k-1$. Take $f \in \mathbb{F}_p[z_1, \ldots, z_k][[x]]$ algebraic over $\mathbb{F}_p(x, z)$ with minimal polynomial P. Chose $\alpha = (\alpha', \alpha_n) \in \mathbb{N}^k$ with $\alpha' \in \mathbb{N}^{k-1}$. Setting $z_k = 0$ in the identity $P(f) = 0$ shows that $f|_{z_k=0} \in \mathbb{F}_p[z_1, \ldots, z_{k-1}][\![x]\!]$ is algebraic over $\mathbb{F}_p(x, z)$. Then the induction hypothesis applies so we know that $(f|_{z_k=0})_{\alpha'} = f_{(\alpha',0)}$ is algebraic over $\mathbb{F}_p(x, z)$. We can apply this argument to the algebraic element $(f - f|_{z_k=0})/z_k$ to get algebraicity of $f_{(\alpha',1)}$ and repeat to show the algebraicity of $f_\alpha = f_{(\alpha',\alpha_n)}$. □

3 The Exponential Differential Equation

In [FH23] the exponential function $\exp_p$ in characteristic p was defined as the xeric solution of $y' = y$. All further solutions of $y' = y$ in $\mathcal{R}_p$ are then given by $\mathcal{C}_p$-multiples of $\exp_p$. In this section we are going to define a different element $\widetilde{\exp}_p \in \mathcal{R}_p$ as an infinite product and show that it is another solution of $y' = y$.

Let us start with an observation. Recall that we introduced w_k as short-hand notation for $w_k := z_1^{p^{k-1}} z_2^{p^{k-2}} \cdots z_{k-1}^p z_k^1$. Then clearly

$$w_k' = \frac{1}{x} z_1^{p^{k-1}-1} z_2^{p^{k-2}-1} \cdots z_{k-1}^{p-1} = \frac{w_k}{x z_1 z_2 \cdots z_k} = \frac{1}{x} w_1^{p-1} w_2^{p-1} \cdots w_{k-1}^{p-1}.$$

Higher derivatives of w_k are in general sums of monomials without any obvious pattern. However, we have the following:

Proposition 3.1. *We have*

$$(x^{p^k} w_k)^{(p^k - p^{k-1}+1)} = -\left(x^{p^{k-1}} w_{k-1}\right)'.$$

This proposition will be a consequence of Theorem 3.4 and Proposition 3.7. One can also give a proof by direct computations.

This property allows for the definition of a solution of $y' = y$ in the following way: A series $y = \sum_{i=0}^\infty a_i(z) x^i \in \mathcal{R}_p$ with $a_i \in \mathbb{F}_p(\!(z_1, z_2, \ldots)\!)$ is a solution of $y' = y$ if and only if

(i) $a_0' = 0$, i.e., $a_0 \in \mathbb{F}_p(z_1^p, z_2^p, \ldots)$, and

(ii) $(a_i(z) x^i)' = a_{i-1}(z) x^{i-1}$.

So we set $a_{p^k-1}(z) := (-1)^k w_k'$ for all k and then define $a_{p^k-m}(z)$ via the equations $x^{p^k-m} a_{p^k-m}(z) = (-1)^k (x^{p^k} w_k)^{(m)}$ for all $m \geq 1$. Proposition 3.1 shows that this is well-defined. This gives rise to a solution of $y' = y$.

This solution will be the solution $\widetilde{\exp}_p$ of $y' = y$ which we will define by different means in the next paragraph. This other definition is less intuitive, but will prove to be more convenient for our calculations. Proposition 3.7 then shows that the two solutions agree.

We define the continuous K-automorphism

$$\sigma : \mathbb{F}_p[\![t]\!] \to \mathbb{F}_p[\![t]\!], \quad t \mapsto \sum_{k=0}^{\infty} t^{p^k},$$

and set recursively

$$g_0 \coloneqq \sigma(x) \quad \text{and} \quad g_{i+1} \coloneqq \sigma(g_i^p z_{i+1}).$$

Further we define

$$H(t) \coloneqq \prod_{k=1}^{p-1} \left(1 - \frac{t}{k}\right)^k \quad \text{for } t \text{ a variable and} \quad \widetilde{\exp}_p \coloneqq \prod_{i=0}^{\infty} H\left((-1)^i g_i\right).$$

Note that $\widetilde{\exp}_p$ is well-defined, as $g_i \in x^{p^i}\mathbb{F}_p[z_1, z_2, \ldots][\![x]\!]$. Clearly $\widetilde{\exp}_p|_{x=0} = 1$. We now show that it is indeed a solution to $y' = y$. The main ingredient is the following Lemma about the logarithmic derivative of H:

Lemma 3.2. *For $s \in \mathcal{R}_p$ we have*

$$\frac{H(s)'}{H(s)} = \frac{s'}{1 - s^{p-1}}.$$

In particular,

$$\frac{H(\sigma(s))'}{H(\sigma(s))} = \frac{s'\sigma(s)}{s} \quad \text{and} \quad \frac{H((-1)^i g_i)'}{H((-1)^i g_i)} = \frac{(-1)^i g_i}{x z_1 \cdots z_i}.$$

Proof. By the additivity of the logarithmic derivative

$$\frac{(fg)'}{fg} = \frac{f'}{f} + \frac{g'}{g},$$

we have

$$\frac{H(s)'}{H(s)} = -\sum_{k=1}^{p-1} \frac{s'}{1 - \frac{1}{k}s}. \tag{2}$$

Set

$$F(t) = \prod_{k=1}^{p-1} \left(t + 1 - \frac{s}{k}\right) = \sum c_i(s) t^i.$$

Using Fermat's Little Theorem we see that $F(t) = (t+1)^{p-1} - s^{p-1}$ as their zero sets agree and in particular, we have $c_0(s) = 1 - s^p$ and $c_1(s) = -1$. Thus, we further obtain, bringing (2) to a common denominator,

$$\frac{H(s)'}{H(s)} = -s'\frac{c_1(s)}{c_0(s)} = -s'\frac{-1}{1 - s^{p-1}} = \frac{s'}{1 - s^{p-1}}.$$

So for $H(\sigma(s))$ we obtain

$$\frac{H(\sigma(s))'}{H(\sigma(s))} = \frac{\sigma(s)'}{1-\sigma(s)^{p-1}} = \frac{s'\sigma(s)}{\sigma(s)-\sigma(s)^p} = \frac{s'\sigma(s)}{s}.$$

In this identity, setting $s = (-1)^i g_{i-1}^p z_i$, i.e., $\sigma(s) = (-1)^i g_i$, we obtain

$$\frac{s'}{s} = \frac{z_i'}{z_i} = \frac{1}{xz_1\cdots z_i}$$

and

$$\frac{H((-1)^i g_i)'}{H((-1)^i g_i)} = \frac{(-1)^i g_i}{xz_1\cdots z_i}.$$

□

Remark 3.3. The identity

$$\frac{H(s)'}{H(s)} = \frac{s'}{1-s^{p-1}}$$

can also be derived from

$$\frac{H(s)'}{H(s)} = -\sum_{k=1}^{p-1}\frac{s'}{1-\frac{1}{k}s} = -s'\cdot\sum_{i=0}^{\infty}s^i\sum_{k=1}^{p-1}\frac{1}{k^i}$$

using the well-known fact

$$\sum_{k=1}^{p-1}k^i \equiv \begin{cases}-1 & \text{if } i\equiv 0 \bmod p-1\\ 0 & \text{else}\end{cases} \bmod p.$$

The formula for the sum of the first k of the i-th powers in terms of Bernoulli numbers is called Faulhaber's formula, who computed the sums for the first 17 values of i in [Fau31] in the early 17-th century. The above-mentioned fact is an easy corollary.

Theorem 3.4. *We have* $\widetilde{\exp}_p' = \widetilde{\exp}_p$.

Proof. By the additivity of the logarithmic derivative and Lemma 3.2 we have

$$\frac{\widetilde{\exp}_p'}{\widetilde{\exp}_p} = \sum_{i=0}^{\infty}\frac{(-1)^i g_i}{xz_1\cdots z_i}.$$

We will show inductively that

$$\sum_{i=0}^{k}\frac{(-1)^i g_i}{xz_1\cdots z_i} = 1 + \frac{(-1)^k g_k^p}{xz_1\cdots z_k}.$$

Then, as $g_k \in x^{p^k}\mathbb{F}_p(z_1,z_2,\ldots)[\![x]\!]$, it follows that

$$\frac{\widetilde{\exp}_p'}{\widetilde{\exp}_p} - 1 = \lim_{k\to\infty}\frac{(-1)^k g_k^p}{xz_1\cdots z_k} \in \bigcap_{k=0}^{\infty} x^{p^k-1}\mathbb{F}_p(z_1,z_2,\ldots)[\![x]\!] = 0$$

and we are done.

The induction is straightforward: For $k = 0$ using that $g_0^p = g_0 - x$ the claim follows immediately. Moreover,

$$1 + \frac{(-1)^k g_k^p}{x z_1 \cdots z_k} + \frac{(-1)^{k+1} g_{k+1}}{x z_1 \cdots z_{k+1}} = 1 + \frac{(-1)^{k+1} g_{k+1}^p}{x z_1 \cdots z_{k+1}}$$

as $g_k^p z_{k+1} = g_{k+1} - g_{k+1}^p$. $\square$

Remark 3.5. This definition of $\widetilde{\exp}_p$ can, in light of this proof, be motivated as follows: We want to find an element of $\mathcal{R}_p$ whose logarithmic derivative is 1. Lemma 3.2 shows that $H(\sigma(x))$ is a good approximation for such an element; its logarithmic derivative is $1 + x^{p-1} + x^{p^2-1} + \ldots \in \frac{1}{x}\mathbb{F}_p[\![x^p]\!]$. By the additivity of the logarithmic derivative, we search for a factor, eliminating the error made, i.e., an element of $\mathcal{R}_p$ whose logarithmic derivative is $-(x^{p-1} + x^{p^2-1} + \ldots)$. For this, choose $s = -g_1$, the primitive of this error series. Then $H(s)$ gives by Lemma 3.2 again a good approximation. Iterating this process we obtain exactly the infinite product defining $\widetilde{\exp}_p$.

Corollary 3.6. *For each $k \in \mathbb{N}$ the series $\pi_k\left(\widetilde{\exp}_p\right)$ is algebraic over $\mathbb{F}_p(x, z_1, \ldots, z_k)$. Further, for each $\alpha \in \mathbb{N}^{(\mathbb{N})}$ the series $\widetilde{\exp}_p \in \mathbb{F}_p(\!(x, z_1, \ldots)\!)$ has an algebraic Laurent series coefficient of z^α in $\mathbb{F}(\!(x)\!)$. The same holds true for $\exp_p$ and any algebraic multiple of it.*

Proof. The series $g_k \in \mathbb{F}_p[z_1, \ldots, z_k][\![x]\!]$ are algebraic. Indeed, g_k satisfies $g_k^p - g_k = g_{k-1}^p z_k$ and by induction and the transitivity of algebraicity, the claim follows. Moreover, $g_k \in z_k\mathbb{F}_p[z_1, \ldots, z_k][\![x]\!]$, so one sees that

$$\prod_{i=0}^{k} H((-1)^i g_i) = \pi_k\left(\widetilde{\exp}_p\right),$$

where the left-hand side is algebraic. Hence all the partial products are algebraic and approximate $\widetilde{\exp}_p$. The rest follows from Proposition 2.4 and Lemma 2.7. $\square$

Write $\widetilde{\exp}_p = \widetilde{e}_0 + \widetilde{e}_1 x + \widetilde{e}_2 x^2 + \ldots$ with $\widetilde{e}_i \in \mathbb{F}_p(z_1, z_2, \ldots)$. The coefficients $\widetilde{e}_i$ of the function $\widetilde{\exp}_p$ have the following remarkable property, which we hinted at the beginning of this section and which already uniquely determines the function $\widetilde{\exp}_p$ as a solution of $y' = y$.

Proposition 3.7. *For all $n \in \mathbb{N}$ we have $\widetilde{e}_{p^n-1} = (-1)^n x w_n'$.*

For the proof we need the following lemma describing certain coefficients of the polynomial $H(s)$:

Lemma 3.8. *Write $H(s) = \sum_{i=0}^{p-1} a_i(s^p)s^i$. Then $a_{p-1} = -1$.*

Proof. By Lemma 3.2 we have

$$H(s) = (1 - s^{p-1})H'(s).$$

Comparing coefficients of powers of s which are congruent to each other modulo p at once, one obtaines the following recursion for the series a_i:

$$a_i = (i+1)a_{i+1} - (i+2)a_{i+2}z^p \quad \text{for } i = 0, \ldots, p-2 \quad \text{and} \quad a_{p-1} = -a_1.$$

From this we obtain that $a_{p-2} = -a_{p-1}$ and inductively, that a_i is divisible by a_{p-1} for all i. It follows that $H(s)$ is divisible by a_{p-1}. Since $H(s)$ does not have a p-fold root, but $a_{p-1} \in \mathbb{F}_p[s^p]$, it follows that $a_{p-1} \in \mathbb{F}_p$ and $a_{p-1} = a_{p-1}(0) = -a_1(0) = -H'(0) = -1$. □

Proof of Proposition 3.7. Denote $h_i := H((-1)^i g_i)$ and write $[x^k]f$ for the coefficient of x^k in the Laurent series expansion of f. We show by induction

$$\widetilde{e}_{p^n-1} = (-1)^m \prod_{i=0}^{m-1} w_i^{p-1} \cdot [x^{p^n-p^m}] \left(\prod_{i=m}^{\infty} h_i \right)$$

for $m = 0, \ldots, n$. For $m = 0$ this is the definition of $\widetilde{e}_{p^n-1}$. For the induction step we need to verify that for all m we have

$$[x^{p^n-p^{m-1}}] \left(\prod_{i=m-1}^{\infty} h_i \right) = -w_{m-1}^{p-1} [x^{p^n-p^m}] \left(\prod_{i=m}^{\infty} h_i \right).$$

Note that $g_i \in \mathbb{F}_p[z_1, z_2, \ldots][\![x^{p^i}]\!]$ and $g_i - w_i x^{p^i} \in \mathbb{F}_p[z_1, z_2, \ldots][\![x^{p^{i+1}}]\!]$. We see $\prod_{i=m}^{\infty} h_i \in \mathbb{F}_p[z_1, z_2, \ldots][\![x^{p^m}]\!]$ and

$$[x^{p^n-p^{m-1}}] \left(\prod_{i=m-1}^{\infty} h_i \right) = \sum_k [x^{kp^m-p^{m-1}}](h_{m-1}) \cdot [x^{p^n-kp^m}] \left(\prod_{i=m}^{\infty} h_i \right). \tag{3}$$

Now one easily checks by induction that

$$h_{m-1} - H((-1)^{m-1} w_{m-1} x^{p^{m-1}}) \in \mathbb{F}_p[z_1, z_2, \ldots][\![x^{p^m}]\!]$$

and we obtain

$$[x^{kp^m-p^{m-1}}](h_{m-1}) = [x^{kp^m-p^{m-1}}](H\left((-1)^{m-1} w_{m-1} x^{p^{m-1}})\right),$$

as the exponents considered are not multiples of p^m. Setting $s = (-1)^{m-1} w_{m-1} x^{p^{m-1}}$ in Lemma 3.8 we can further compute

$$[x^{kp^m-p^{m-1}}](H\left((-1)^{m-1} w_{m-1} x^{p^{m-1}})\right) = \begin{cases} -((-1)^{m-1} w_{m-1})^{p-1} & \text{if } k = 1 \\ 0 & \text{otherwise,} \end{cases}$$

which shows that the sum on the right-hand side of (3) only has one non-trivial summand, namely for $k = 1$. This finishes the induction step. Now setting $m = n$ we obtain

$$\widetilde{e}_{p^n-1} = (-1)^n \prod_{i=0}^{n-1} w_i^{p-1} = (-1)^n x w_n'.$$

□

4 The p^k-curvatures

Consider the equation

$$y' + ay = 0.$$

for $a \in \mathbb{F}_p[z_1, z_2, \ldots, z_k]((x))$. Write $L = \partial + a$ for the corresponding differential operator and assume that its initial form L_0 is an element of $\mathbb{F}_p[x][\partial]$, i.e., $a = \frac{\rho}{x} + \widetilde{a}$, where $\rho \in \mathbb{F}_p$ and $\widetilde{a} \in \mathbb{F}_p[z_1, \ldots, z_k][\![x]\!]$. Without loss of generality we may assume that its local exponent is given by $\rho = 0$. Indeed, replacing y by $x^{-\rho}y$ the differential equation changes to

$$y' + \left(a - \frac{\rho}{x}\right) y = 0,$$

whose local exponent is 0.

Recall that the elements $\widetilde{e}_i \in \mathbb{F}_p[z_1, z_2, \ldots]$ were defined as the coefficients of the solution $\widetilde{\exp}_p = \widetilde{e}_0 + \widetilde{e}_1 x + \widetilde{e}_2 x^2 + \ldots$ of the exponential differential equation $y' = y$ in Section 3. In particular, this means that $(\widetilde{e}_i x^i)' = \widetilde{e}_{i-1} x^{i-1}$.

Lemma 4.1. *The series*

$$y_0 := \sum_{i=0}^{\infty} (-1)^i \widetilde{e}_i x^i L^i(1) \tag{4}$$

is an element of $\mathcal{R}_p$ and a solution of $Ly = 0$.

Proof. The first assertion is trivial, as $\operatorname{ord}_x \widetilde{e}_i x^i L^i(1) \geq i$ and thus the sum is well-defined.

For the second part we compute using the product rule

$$Ly_0 = \sum_{i=1}^{\infty} (-1)^i \widetilde{e}_{i-1} x^{i-1} L^i(1) + \sum_{i=0}^{\infty} (-1)^i \widetilde{e}_i x^i L^{i+1}(1) = 0.$$

□

Remark 4.2. This Lemma gives another verification of the fact that any differential equation of order one has a solution in $\mathcal{R}_p$. Moreover, it extends to systems of higher dimensions, generalizing a proof of Cartier's Lemma (cf. Proposition 4.5) given by Chambert-Loir [Cha02, p.184]:

Let $Y' = AY$ be a system of first order differential equations. Define the matrices A_k via $A_{k+1} = A_k' + A_k \cdot A$. Then for any solution Y of the system, we have $Y^{(n)} = A_n Y$ and the same computation as in the proof above shows that

$$Y = \sum_{i=0}^{\infty} (-1)^i \widetilde{e}_i A_i$$

is a fundamental matrix of solutions of $Y' = AY$.

Thereby we obtain – modulo the equivalence of systems of differential equations of order 1 and scalar equations of higher order – another verification of the existence of a full basis of solutions of any regular singular differential equation in $\mathcal{R}_p$.

In analogy to the p-curvature we define the p^k-*curvature* as the operator

$$L^{p^k} : \mathcal{R}_p \to \mathcal{R}_p.$$

Lemma 4.3. *The p^k-curvature of $L = \partial + a$ is an $\mathcal{R}_p^{(k-1)} = \mathbb{F}_p(z_1, z_2, \ldots z_{k-1})(\!(x)\!)$-linear map. Consequently, on $\mathcal{R}_p^{(k-1)}$ it is given by the value of 1: $L^{p^k}(1) =: a_{p^k}$.*

Proof. By induction one easily shows for any $f, g \in \mathbb{F}_p(z_1, z_2, \ldots z_{k-1})(\!(x)\!)$ the equation

$$L^m(fg) = \sum_{j=0}^{\infty} \binom{m}{j} L^j(f)\partial^{m-j} g$$

holds true. In particular, for $m = p^k$ only two of the binomial coefficients do not vanish modulo p and we obtain

$$L^{p^k}(fg) = gL^{p^k}(f) + f\partial^{p^k} g = gL^{p^k}(f),$$

as $\partial^{p^k}(\mathbb{F}_p(z_1, z_2, \ldots z_{k-1})(\!(x)\!)) = 0$. $\square$

We will therefore by abuse of notation also call a_{p^k} the p^k-curvature of $L = \partial + a(x)$.

Proposition 4.4. *The p^k-curvatures a_{p^k} have the alternate characterization:*

$$a_{p^k} = \sum_{i=0}^{k} \left(a^{(p^i-1)}\right)^{p^{k-i}} = \left(a_{p^{k-1}}\right)^p + a^{(p^k-1)}.$$

Proof. Write a_m for $(\partial + a)^m(1)$. First note that a_m can be written as

$$a_m = \sum_{\alpha \in \mathcal{A}_m} \lambda_\alpha \prod_{j=0}^{m-1} \left(a^{(j)}\right)^{\alpha_j}, \tag{5}$$

where

$$\mathcal{A}_m := \left\{ \alpha = (\alpha_0, \ldots \alpha_{m-1}) \in \mathbb{N}^m : \sum_{j=0}^{m-1} \alpha_j (j+1) = m \right\}.$$

Indeed, $(\partial + a)(1) = a$ and inductively, for each summand in (5) both, multiplication by a, and differentiation, give a monomial with exponents in $\mathcal{A}_{m+1}$.

Next, we show that each of the coefficients λ_α for $\alpha \in \mathcal{A}_m$ in the expansion (5) of a_m is given by:

$$\begin{aligned} \lambda_\alpha &= \frac{m!}{\prod_{j=0}^{m-1} \alpha_j!((j+1)!)^{\alpha_j}} \\ &= \binom{m}{\alpha_0, 2\alpha_1, \ldots, m\alpha_{m-1}} \prod_{j=0}^{m-1} \binom{\alpha_j(j+1)}{j+1, \ldots, j+1} \frac{1}{\alpha_j!}. \end{aligned} \tag{6}$$

Note that the right-hand side of this equation is an integer and can be reduced properly modulo any prime p. Again, we proceed by induction. For $m = 0$, $\mathcal{A}_0 = \{()\}$ and $\lambda_0 = 1$. Denote by ε_k for $0 \le k \le m$ the element $(0, \dots, 0, 1, 0, \dots, 0) \in \mathbb{N}^{m+1}$, where the entry 1 is in the $k+1$-st position. We embed $\mathbb{N}^m$ in $\mathbb{N}^{m+1}$ by $\mathbb{N}^m \cong \mathbb{N}^m \times \{0\} \subseteq \mathbb{N}^{m+1}$. Then for $\alpha \in \mathcal{A}_{m+1}$ we get

$$\begin{aligned}\lambda_\alpha &= \lambda_{\alpha-\varepsilon_0} + \sum_{j=0}^{m-1} (\alpha_j + 1)\lambda_{\alpha+\varepsilon_j-\varepsilon_{j+1}} \\ &= \frac{(m+1)!}{\prod_{j=0}^m \alpha_j!((j+1)!)^{\alpha_j}} \left(\frac{\alpha_0 \cdot 1!}{m+1} + \sum_{j=0}^{m} \frac{(j+2)\alpha_{j+1}}{m+1} \right) = \frac{(m+1)!}{\prod_{j=0}^m \alpha_j!((j+1)!)^{\alpha_j}}\end{aligned}$$

using the induction hypothesis for $\alpha + \varepsilon_j - \varepsilon_{j+1} \in \mathcal{A}_m$.

Now we show that for $m = p^k$ only a small portion of the coefficients λ_α are non-zero, namely only if $\alpha = p^{k-\ell}\varepsilon_{p^\ell-1}$ for $\ell = 0, 1, \dots, k$. In these cases $\lambda_\alpha = 1$.

By Lucas' Theorem applied to the left multinomial coefficient in (6), it follows that $\lambda_\alpha = 0$ except $p^k = m = \alpha_{j_0}(j_0 + 1)$ for some j_0. Consequently $j_0 = p^\ell - 1$ and $\alpha_{j_0} = p^{k-\ell}$ for some ℓ. This means $\alpha = p^{k-\ell}\varepsilon_{p^\ell-1}$. We compute, splitting the multinomial coefficient into a product of binomial coefficients, accounting for one factor of $p^{k-\ell}!$ in each of these binomials and using Lucas' Theorem:

$$\lambda_\alpha = 1 \cdot \frac{1}{p^{k-\ell}!}\binom{p^k}{p^\ell, \dots, p^\ell} = \prod_{j=0}^{p^{k-\ell}} \binom{jp^\ell - 1}{p^\ell - 1} = \prod_{j=0}^{p^{k-\ell}} \binom{j-1}{0} = 1.$$

This finishes the proof. □

This is a generalization of the formula $a_p = a^p + a^{(p-1)}$ for the p-curvature for first order equations [BCR23, Thm. 3.12]. It has no obvious extension to higher dimensional differential equations, or, equivalently, larger systems of first order differential equations.

If $a \in \mathbb{F}_p(\!(x)\!)$, then $a_{p^k} = a_p^{p^{k-1}}$, i.e., the evaluation of L^{p^k} at 1 is just the p^{k-1}-st power of the evaluation of L^p at 0. Therefore it vanishes, if and only if the p-curvature does so. However, if $a \in \mathbb{F}_p(z_1, \dots z_k)(\!(x)\!)$ the p^j-curvatures for $j = 1, \dots k+1$ are not just powers of each other, but $a_{p^j} = a_{p^{k+1}}^{j-k}$ for $j > k$. If a is any element in $\mathcal{R}_p$, there need not be any such relation between the p^j-curvatures.

The following proposition shows that the p^k-curvatures generalize the p-curvature. Recall that we write $\mathcal{R}_p^{(k)} \subseteq \mathcal{R}_p$ for $\mathbb{F}_p(z_1, \dots, z_k)(\!(x)\!)$. It generalizes Cartier's Lemma relating the vanishing of the p-curavture to the existence of solutions for differential equations in positive characteristic p, first appearing in [Kat72] and being reproduced in [BCR23, Thm. 3.18].

Proposition 4.5 (Extension of Cartier's Lemma)**.** *Let $L = \partial + a$ be a differential operator with regular singularity at 0 and local exponent $\rho = 0$.*

(a) If $a \in \mathbb{F}_p(x, z_1, \dots, z_k) \subseteq \mathcal{R}_p^{(k)}$ is a rational function, then the following are equivalent:

(i) The differential equation $Ly = 0$ has a non-zero solution in $\mathbb{F}_p(z_1, \ldots, z_k, x)$.

(ii) The p^{k+1}-curvature of L vanishes, i.e., $a_{p^{k+1}} = 0$.

(iii) The operator $\partial^{p^{k+1}}$ is divisible by L in $\mathbb{F}_p(z_1, \ldots, z_k, x)[\partial]$.

(b) If $a \in \mathcal{R}_p^{(k)}$ is algebraic over $\mathbb{F}_p(x, z_1, \ldots, z_k)$, then the following are equivalent:

(i) The differential equation $Ly = 0$ has a non-zero algebraic solution in $\mathcal{R}_p^{(k)}$.

(ii) The p^{k+1}-curvature of L vanishes, i.e., $a_{p^{k+1}} = 0$.

(iii) The operator $\partial^{p^{k+1}}$ is divisible by L in $\mathcal{R}_{p,alg}^{(k)}[\partial]$, where $\mathcal{R}_{p,alg}^{(k)}$ denotes the algebraic elements of $\mathcal{R}_p^{(k)}$.

(c) If $a \in \mathcal{R}_p^{(k)}$ is arbitrary, then the following are equivalent:

(i) The differential equation $Ly = 0$ has a non-zero solution in $\mathcal{R}_p^{(k)}$.

(ii) The p^{k+1}-curvature of L vanishes, i.e., $a_{p^{k+1}} = 0$.

(iii) The operator $\partial^{p^{k+1}}$ is divisible by L in $\mathcal{R}_p^{(k)}[\partial]$.

Proof. The proof for all three assertions works analogously, we state the proof for (a) here.

Assume (i) holds, i.e., there is a solution $y \in \mathbb{F}_p(x, z_1, \ldots, z_k)$ of $Ly = 0$. Then

$$a_{p^{k+1}} = L^{p^{k+1}}(1) = y^{-1} L^{p^{k+1}}(y) = 0,$$

which shows (ii). For the converse implication consider the solution y of $Ly = 0$ given in Lemma 4.1, Equation (4). As $L^{p^{k+1}}(1) = 0$ this is a finite sum of rational functions in the variables $x, z_1, \ldots, z_k$, which proves (i).

For the equivalence of (i) and (iii) we use that $\mathbb{F}_p(x, z_1, \ldots, z_k)[\partial]$ is a (left- and right-) Euclidean ring. In fact, the skew-polynomial ring over any (skew-) field is Euclidean, as observed by Ore [Ore33]. Thus we may write $\partial^{p^{k+1}} = QL + R$ for some differential operator R, Q, where $\operatorname{ord} R < \operatorname{ord} L = 1$. If $y \in \mathbb{F}_p(x, z_1, \ldots, z_k)$ is a solution of $Ly = 0$, then $Ry = \partial^{p^{k+1}} y - QLy = 0$ and consequently the solution of $Ly = 0$ is also a solution of $Ry = 0$. Because R is of order 0, this means $R = 0$ and thus (iii) follows.

Conversely, assume that $\partial^{p^{k+1}} = QL$. The kernel of $\partial^{p^{k+1}}$ is $\mathbb{F}_p(x, z_1, \ldots, z_k)$ and thus a p^{k+1}-dimensional $\mathbb{F}_p(x^p, z_1^p, \ldots, z_k^p)$-vector space. The $\mathbb{F}_p(x^p, z_1^p, \ldots, z_k^p)$-dimensions of the kernels of Q respectively L in $\mathbb{F}_p(x, z_1, \ldots, z_k)$ are at most $p^{k+1} - 1$ respectively 1, thus equality must hold. In particular the kernel of L is not empty, i.e., (i) holds. $\square$

5 Product Formulas and Algebraicity for Solutions of Equations of Order 1

The goal of this section is to generalize the product formula for $\exp_p$ developed in Section 3 to solutions of arbitrary first order differential equations.

Recall that we defined $w_k := z_1^{p^{k-1}} z_2^{p^{k-2}} \cdots z_k^1$ and $\mathcal{S}_p$ as the completion of $\mathbb{F}_p[x, x^p w_1, x^{p^2} w_2, x^{p^3} w_3 \ldots]$ in $\mathcal{R}_p$, as well as $\mathcal{S}_p^{(k)} := \mathcal{S}_p \cap \mathcal{R}_p^{(k)}$. Further recall the projection maps $\pi_k : \mathcal{R}_p \to \mathcal{R}_p^{(k)}$, taking the constant term of an element with respect to $z_{k+1}, z_{k+2}, \ldots$.

Theorem 5.1. *Let $L = \partial + a$ be a first order linear differential operator with rational function coefficient $a \in \mathbb{F}_p(x)$ (or algebraic coefficient $a \in \mathbb{F}_p(\!(x)\!)$), regular singularity at 0 and local exponent $\rho = 0$. Then, for every $k \in \mathbb{N}$ there exists a series $h_k \in 1 + x^{p^k} w_k \mathcal{S}_p^{(k)}$, which is algebraic over $\mathbb{F}_p(x, z_1, z_2, \ldots, z_k)$, such that $h := \prod_{k=0}^{\infty} h_k$ satisfies $Lh = 0$. In particular, $\pi_i(h) = \prod_{j=0}^{i} h_j$ is algebraic over $\mathbb{F}_p(x, z_1, \ldots, z_i)$ for all i.*

With Proposition 2.4 and Lemma 2.7, this has the following immediate consequences:

Corollary 5.2. *Let $L = \partial + a$ be a first order differential operator with local exponent $\rho \in \mathbb{F}_p$. Then its xeric solution has algebraic projections.*

Corollary 5.3. *Let y be the solution h of $Ly = 0$ defined in Theorem 5.1 or the xeric solution of this equation. Then for any $\alpha \in \mathbb{N}^{(\mathbb{N})}$ the coefficient $y_\alpha \in \mathbb{F}_p(\!(x)\!)$ of z^α in y is algebraic over $\mathbb{F}_p(x)$. In particular, the initial series $y|_{z_1 = z_2 = \cdots = 0}$ of y is algebraic.*

For the proof of Theorem 5.1 we need the following lemma:

Lemma 5.4. *Let $L = \partial + a$ be a differential operator with $a \in \mathcal{S}_p^{(i)}$. There is an algebraic element $v \in \mathcal{S}_p^{(i)}$, such that the p^{i+1}-curvature of the modified operator*

$$L_{\leq i} := L - v^p (x^{p^{i+1}} w_{i+1})' = \partial + a - v^p (x^{p^{i+1}} w_{i+1})'$$

vanishes.

Proof. We write $a = c(w_{i+1} x^{p^{i+1}})' + r$ in the $\mathcal{C} \cap \mathcal{R}_p^{(i)}$-vector space $\mathcal{R}_p^{(i)}$ for some $c \in \mathcal{C} \cap \mathcal{R}_p^{(i)}$ and some r in the direct complement of $(w_{i+1} x^{p^{i+1}})'$. Then one checks that $(r)^{(p^{i+1}-1)} = 0$, cf. [FH23, Lem. 3.4]. By Proposition 4.4 the p^{i+1}-curvature of $L_{\leq i}$ is given by

$$\begin{aligned} L_{\leq i}^{p^{i+1}}(1) &= (L_{\leq i}^{p^i}(1))^p + (a - v^p (x^{p^{i+1}} w_{i+1})')^{(p^{i+1}-1)} \\ &= (L_{\leq i}^{p^i}(1))^p + (a)^{(p^{i+1}-1)} - v^p (x^{p^{i+1}} w_{i+1})^{(p^{i+1})} \\ &= (L_{\leq i}^{p^i}(1))^p + (-1)^{i+1} \widetilde{a}^p - v^p (x^{p^{i+1}} w_{i+1})^{(p^{i+1})} \\ &= (L_{\leq i}^{p^i}(1))^p + (-1)^{i+1} (\widetilde{a} - v)^p, \end{aligned}$$

where we have used that $(x^{p^j} w_j)^{(p^j)} = (-1)^j$ for all j.

The vanishing of the p-curvature of L_k is thus equivalent to

$$(-1)^{i+1} L_{\leq i}^{p^i}(1) + \widetilde{a} = v$$

The left hand side of this equation can be expanded as a polynomial in v^p with algebraic coefficients in $\mathcal{S}_p^{(i)}$. Thus we can invoke the implicit function theorem to find an algebraic solution v. □

Proof of Theorem 5.1. For $i \in \mathbb{N}$ we will construct a sequence of operators $L_{\leq i}$, modifications of L, such that their p^{i+1}-curvature vanishes, converging to L in the x-adic topology. Further we will construct algebraic $h_i \in \mathcal{S}_p^{(i)}$ with $\pi_{i-1}(h_i) = 1$ in such a way that $b_i := \prod_{j=0}^{i} h_j$ satisfies $L_{\leq i}(b_i) = 0$. Then, we will show that the sequence b_i converges to a solution h of $Ly = 0$.

We set $V_i := v_i^p (x^{p^{i+1}} w_{i+1})'$ and

$$L_{\leq i} := L - V_i,$$

where $v_i \in \mathcal{S}_p^{(i)}$ is algebraic and chosen in accordance with Lemma 5.4 and such that the p^{i+1}-curvature of $L_{\leq i}$ vanishes. Denote by t_i the solution of $L_{\leq i} y = 0$, obtained from Lemma 4.1, i.e.,

$$t_i := \sum_{j=0}^{p^{i+1}-1} (-1)^j \widetilde{e}_j L_{\leq i}^j(1) x^j.$$

Note that the sum is finite, because $L_{\leq i}^{p^{i+1}}(1) = 0$ and t_i is algebraic over $\mathbb{F}_p(x, z_1, \ldots, z_i)$, as $L_{\leq i}$ has algebraic coefficients.

Denote by $L_{>i}$ the differential operator $\partial + V_i$. Rewriting with shift 0, we obtain $x\partial + xV_i = S_{>i} + T_{>i}$ in the language of Fuchs' Theorem in positive characteristic, Theorem 2.1. By construction the composition $S_{>0}T_{>0}$ maps $\mathcal{S}_p$ to $w_{i+1}\mathcal{S}_p$. So the equation $L_{>i} y = 0$ has a solution $q_i \in \mathcal{S}_p$ with $\pi_i(q_i) = 1$. We show that if $f_1 \neq 0$ is a solution of $L_{\leq i} y = 0$, then $L(f_1 f_2) = 0$ if and only if $L_{>i}(f_2) = 0$. Indeed,

$$\begin{aligned} L(f_1 f_2) &= (\partial + a(x) - V_i + V_i)(f_1 f_2) \\ &= f_2 \partial f_1 + (a(x) - V_i) \cdot f_1 f_2 + f_1 \partial f_2 + V_i \cdot f_1 f_2 \\ &= f_2 \cdot L_{\leq i}(f_1) + f_1 \cdot L_{>i}(f_2) = f_1 \cdot L_{>i}(f_2) \end{aligned}$$

Set $h_0 := t_0$. Then $L_{\leq 0}(b_0) = L_{\leq 0}(h_0) = 0$ by definition. Now assume that we have constructed $h_0, \ldots, h_i$ already.

We set

$$u_{i+1} := t_{i+1}^{-1} q_{i+1}^{-1} b_i q_i.$$

Then, $u_{i+1} \in \mathcal{C}_p$, as according to the observation from above, both $t_{i+1}q_{i+1}$ and $b_i q_i$ are solutions of the first order equation $Ly = 0$. Note that $\pi_i(u_{i+1}) = \pi_i(t_{i+1})^{-1} b_i$

is algebraic, as both factors are.

Further, we define

$$h_{i+1} \coloneqq \pi_i(u_{i+1})t_{i+1}b_i^{-1} \in \mathcal{S}_p^{(i+1)},$$

which is algebraic, since all the factors are. Moreover,

$$L_{\leq i+1}(b_{i+1}) = L_{\leq i+1}(\pi_i(u_{i+1})t_{i+1}) = \pi_i(u_{i+1})L_{\leq i+1}(t_{i+1}) = 0,$$

since we already established $u_{i+1} \in \mathcal{C}_p$ and

$$\pi_i(h_{i+1}) = \pi_i(u_{i+1}t_{i+1}b_i^{-1}) = \pi_i(q_i)\pi_i(q_{i+1})^{-1} = 1.$$

Thus, h_{i+1} has the required properties.

Finally, we have to show that b_i converges to a solution of $Ly = 0$ as $i \to \infty$. First, note that $h \coloneqq \prod_{i=0}^{\infty} h_i$ is well-defined, as $\operatorname{ord}_x(h_i - 1) \geq p^{i+1}$. We prove that $[x^n]Lh = 0$ for all $n \in \mathbb{N}$. Let n be fixed and choose m with $n + 1 < p^{m+1}$. Then $\operatorname{ord}_x(h - b_m) \geq p^{m+1} > n + 1$ and consequently

$$\operatorname{ord}_x(L(h - b_m)) > n.$$

Clearly $\operatorname{ord}_x(V_m b_m) > n$ and therefore

$$[x^n]Lh = [x^n]Lb_m = [x^n]L_{\leq m}b_m + [x^n]V_m b_m = 0.$$

$\square$

Remark 5.5. We can replace a by an algebraic element of $\mathcal{S}_p^{(k)}$ in the theorem above. For the proof one sets

$$h_i \coloneqq \pi_{i-1}(t_k)^{-1}\pi_i(t_k)$$

for $1 \leq i \leq k$ and $h_0 = \pi_0(t_k)$. Since t_k is algebraic so are $h_0, \dots, h_k$ and

$$\pi_{i-1}(h_i) = \pi_{i-1}(\pi_{i-1}(t_k)^{-1}\pi_i(t_k)) = 1.$$

For h_i with $i > k$ one proceeds as in the proof of the theorem.

Example 5.6. In the case of the exponential differential equation $y' = y$ we have $a = -1$. In this case the equation for v_0 reads according to Lemma 5.4

$$-1 - v_0^p x^{p-1} + v_0 = 0,$$

which has the solution $v_0 = x^{-1}\sigma(x)$, where $\sigma(x) = \sum_{k=0}^{\infty} x^{p^k}$. The differential equation $L_{\leq 0}$ then reads $xy' = \sigma(x)y$, or equivalently:

$$\frac{y'}{y} = \frac{\sigma(x)}{x}.$$

By Lemma 3.2 we have

$$\frac{(H(\sigma(x)))'}{H(\sigma(x))} = \frac{\sigma(x)}{x},$$

which shows that shows that

$$H(\sigma(x)) = \prod_{k=1}^{p-1} \left(1 - \frac{1}{k}\sigma(x)\right)^k =: h_0$$

solves the equation $L_{\leq 0}y = 0$. So we recover the beginning of the infinite product defining $\widetilde{\exp}_p$.

6 Trigonometric functions

Having investigated the exponential function in positive characteristic one cannot resist to look also at the sine and cosine function, i.e., at the solutions of the second order differential equation

$$y'' + y = 0.$$

The local exponents at 0 are 0 and 1. The equation has a 2-dimensional solution space over the field of constants $\mathcal{C}_p = \mathbb{F}_p(z_1^p, z_2^p, \dots)((x^p))$. Here are, for $p = 3$, the two xeric solutions, which we call $\sin_p$ and $\cos_p$:

$$\sin_3(x) = x + z_1x^3 + z_1x^5 + (z_1^2 + z_1)x^7 + z_1^3z_2x^9 + (z_1^3z_2 + 2z_1)x^{11} + \dots$$

$$\cos_3(x) = 1 + x^2 + 2z_1x^4 + (z_1^2 + 2z_1)x^6 + (z_1^2 + 2z_1 + 2)x^8 + (2z_1^3z_2 + z_1)x^{10} + \dots$$

In this situation, it is tempting to expect again an algebraic relation of the form $\sin_p^2 + \cos_p^2 = 1$ as in the characteristic zero case. This can easily be disproved, and it is also not clear a priori how $\sin_p$ and $\cos_p$ relate to $\exp_p$. To explore these questions, expand

$$\exp_p = e_0 + e_1x + e_2x^2 + \dots$$

with $e_i \in \mathbb{F}_p[z]$ and split $\exp_p$ into

$$\mathrm{even}(\exp_p) = e_0 + e_2x^2 + e_4x^4 + \dots,$$

$$\mathrm{odd}(\exp_p) = e_1x + e_3x^3 + e_5x^5 + \dots$$

as series of even and odd degrees. Clearly, both series are solutions of $y'' - y = 0$ since $(e_ix^i)'' = e_{i-2}x^{i-2}$. They are, however, not xeric . Let us denote by $\sinh_p$ and $\cosh_p$ the xeric solutions of $y'' - y = 0$. Further, for $\mathrm{char}(K) > 2$, it is immediate that

$$\mathrm{even}(\exp_p) = \frac{1}{2}\left(\exp_p(z, x) + \exp_p(z, -x)\right),$$

$$\mathrm{odd}(\exp_p) = \frac{1}{2}\left(\exp_p(z, x) - \exp_p(z, -x)\right).$$

This proves by Corollary 3.6 that both $\mathrm{even}(\exp_p)$ and $\mathrm{odd}(\exp_p)$ have algebraic projections: they play the role of the classical hyperbolic sine and cosine functions sinh and cosh in characteristic p. By Proposition 2.4, the corresponding xeric solutions, $\sinh_p$ and $\cosh_p$, also have algebraic projections.

The same argument applies for the equation $y'' + y = 0$ and $\mathrm{char}(K) > 2$. The two series

$$\frac{1}{2}\left(\exp_p(z, ix) + \exp_p(z, -ix)\right),$$
$$\frac{1}{2}\left(\exp_p(z, ix) - \exp_p(z, -ix)\right),$$

where $i \in \overline{\mathbb{F}}_p$ is a square root of -1, form a basis of solutions. This proves:

Proposition 6.1. *The projections of* $\cosh_p$, $\sinh_p$, $\cos_p$, $\sin_p$ *are all algebraic.*

The next observation is somewhat more surprising.

Proposition 6.2. *Let* $\sinh_p$ *and* $\cosh_p$ *denote the xeric solutions of* $y'' - y = 0$ *with respect to the local exponents* $\rho_1 = 1$ *and* $\rho_2 = 1$. *Then the following identity holds,*

$$\exp_p = \cosh_p + \frac{1}{1 - \sigma^p} \sinh_p,$$

where $\sigma(x) = x + x^p + x^{p^2} + \dots$

Remark 6.3. In this formula, there is an asymmetry between $\sinh_p$ and $\cosh_p$. On the other hand, by definition the symmetric formula $\exp_p = \mathrm{even}(\exp_p) + \mathrm{odd}(\exp_p)$ holds.

Proof. The functions $\sinh_p$ and $\cosh_p$, as xeric solutions, are uniquely determined as the solutions of $y'' = y$ with $\langle \sinh_p \rangle_{0,0} = 0, \langle \sinh_p \rangle_{1,0} = 1$ respectively $\langle \cosh_p \rangle_{0,0} = 1, \langle \cosh_p \rangle_{1,0} = 0$.

Write even_p and odd_p for $\mathrm{even}(\exp_p)$ and $\mathrm{odd}(\exp_p)$. Note that $\langle \exp_p \rangle_{0,0} = 1$ and consequently $\langle \mathrm{odd}_p \rangle_{0,0} = 0$ and $\langle \mathrm{even}_p \rangle_{0,0} = 1$. A short computation shows that

$$\sinh_p = \mathrm{odd}_p \cdot \frac{x}{\langle \mathrm{odd}_p \rangle_{1,0}}$$

and

$$\cosh_p = \mathrm{even}_p - \sinh_p \frac{\langle \mathrm{even}_p \rangle_{1,0}}{x} = \mathrm{even}_p - \mathrm{odd}_p \frac{\langle \mathrm{even}_p \rangle_{1,0}}{\langle \mathrm{odd}_p \rangle_{1,0}},$$

since for example

$$\left\langle \mathrm{odd}_p \cdot \frac{x}{\langle \mathrm{odd}_p \rangle_{1,0}} \right\rangle_{1,0} = \langle \mathrm{odd}_p \rangle_{1,0} \cdot \frac{x}{\langle \mathrm{odd}_p \rangle_{1,0}} = x.$$

hold. As $\exp_p = \mathrm{odd}_p + \mathrm{even}_p$ we obtain

$$\exp_p = \cosh_p + \frac{1}{x} \sinh_p \cdot \langle \mathrm{odd}_p + \mathrm{even}_p \rangle_{1,0}.$$

We set

$$K := \frac{1}{x} \langle \mathrm{odd}_p + \mathrm{even}_p \rangle_{1,0} = \frac{1}{x} \langle \exp_p \rangle_{1,0}.$$

and we are left to show that $K = \dfrac{1}{1 - \sigma^p}$.

Recall the infinite product decomposition of the solution $\widetilde{\exp}_p = h_0 \cdot h_1 \cdot h_2 \cdots$ of $y' = y$, where $h_i = H((-1)^i g_i)$, the g_i are defined recursively and H is a polynomial satisfying $H(s) = (1 - s^{p-1})H'(s)$, see Lemma 3.2. Then $\widetilde{\exp}_p = \exp_p\langle\widetilde{\exp}_p\rangle_{0,0}$ and substituting in the definition of K we obtain

$$xK\langle\widetilde{\exp}_p\rangle_{0,0} = \langle\widetilde{\exp}_p\rangle_{1,0}.$$

We have the equality

$$\langle\widetilde{\exp}_p\rangle_{0,0} = \langle h_0\rangle_{0,0} \cdot \langle h_0^{-1}\widetilde{\exp}_p\rangle_{0,0}.$$

Indeed, $h_0^{-1}\widetilde{\exp}_p = h_1 \cdot h_2 \cdots$ is a series in x^p with coefficients in $\mathbb{F}_p(z_1, z_2, \ldots)$ and such that $\pi_0\big(h_0^{-1}\widetilde{\exp}_p\big) = 1$. Moreover, h_0 is a series in x only. Therefore a monomial in $\langle\widetilde{\exp}_p\rangle_{0,0}$ can be written uniquely as a product of a monomial in $\langle h_0\rangle_{0,0}$ and a monomial in $\langle h_0^{-1}\widetilde{\exp}_p\rangle_{0,0}$.

Similarly we obtain

$$\langle\widetilde{\exp}_p\rangle_{1,0} = \langle h_0\rangle_{1,0} \cdot \langle h_0^{-1}\widetilde{\exp}_p\rangle_{0,0}$$

and consequently

$$xK\langle\widetilde{\exp}_p\rangle_{0,0} = \langle h_0\rangle_{1,0} \cdot \langle h_0^{-1}\widetilde{\exp}_p\rangle_{0,0} = \langle h_0\rangle_{1,0} \cdot \langle\widetilde{\exp}_p\rangle_{0,0} \cdot \langle h_0\rangle_{0,0}^{-1}.$$

Using that $\langle h_0\rangle_{1,0} = x\langle h_0'\rangle_{0,0}$ we get $K = \langle h_0'\rangle_{0,0}\langle h_0\rangle_{0,0}^{-1}$. Recall that $h_0 = H(\sigma)$ and thus, by Lemma 3.2 and the identity $\sigma - \sigma^p = x$ we have

$$\frac{h_0'}{h_0} = \frac{\sigma}{\sigma - \sigma^p} = \frac{\sigma}{x}.$$

Moreover,

$$\langle h_0'\rangle_{0,0} = \left\langle \frac{h_0\sigma}{x} \right\rangle_{0,0} = \langle h_0\rangle_{0,0} + \sigma^p \left\langle \frac{h_0}{x} \right\rangle_{0,0} = \langle h_0\rangle_{0,0} + \sigma^p\langle h_0'\rangle_{0,0}$$

and thus $K = \dfrac{1}{1 - \sigma^p}$. □

The considerations in this chapter motivate to investigate the following two problems. Firstly, the concept of xeric series is intended to select among the numerous solutions of a differential equation some "distinguished" and hence unique ones. However, the choice of this basis is not as "natural" as one could hope for, a testimony of which is the asymmetric formula in Proposition 6.2. Also the exponential function $\widetilde{\exp}_p$, characterized by Proposition 3.7, and the solutions $\mathrm{even}(\exp_p)$ and $\mathrm{odd}(\exp_p)$ of the differential equation $y + y'' = 0$ have noticeable properties among the solutions of their respective equations. So the quest for truly distinguished and natural solutions remains open.

Secondly, Problem 1.1, or, equivalently, Problem 2.6, has been solved for the second order differential equations $y'' = y$ and $y'' = -y$ in this last section, proving the desired algebraicity. For arbitrary linear differential equations of order greater than or equal to two it is still unclear if the projections of suitably chosen solutions are again algebraic.

References

[BCR23] A. Bostan, X. Caruso, and J. Roques. "Algebraic Solutions of Linear Differential Equations: An Arithmetic Approach" (2023). DOI: 10.48550/arXiv.2304.05061.

[Bou97] A. Boulanger. "Contribution à l'étude des équations différentielles linéaires et homogènes intégrables algébriquement". Paris: École polytechnique, 1897. 122 pp.

[Cha02] A. Chambert-Loir. "Théorèmes d'algébricité en géométrie diophantienne". *Astérisque*. Séminaire Bourbaki, exp. no 886 282 (2002), pp. 175–209.

[Dwo90] B. Dwork. "Differential Operators with Nilpotent p-Curvature". *American Journal of Mathematics* 112.5 (Oct. 1990), pp. 749–786. DOI: 10.2307/2374806.

[Fau31] J. Faulhaber. *Academia Algebrae: darinnen dir miraculosische Guvontiones zu den höchsten Costen weiters continuirt u. profitiert werden.* Augspurg: Johann Remmelin, 1631. 88 pp.

[FH23] F. Fürnsinn and H. Hauser. "Fuchs' Theorem on Linear Differential Equations in Arbitrary Characteristic" (2023). DOI: 10.48550/arXiv.2307.01712.

[Hon81] T. Honda. "Algebraic Differential Equations". *Symposia Mathematic.* Vol. XXIV (Sym- pos., INDAM, Rome, 1979). London-New York: Academic Press, 1981, pp. 169–204.

[Kat72] N. M. Katz. "Algebraic Solutions of Differential Equations (p-Curvature and the Hodge Filtration)". *Inventiones Mathematicae* 18.1-2 (Mar. 1972), pp. 1–118. DOI: 10.1007/BF01389714.

[Ore33] O. Ore. "Theory of Non-Commutative Polynomials". *The Annals of Mathematics* 34.3 (July 1933), p. 480. DOI: 10.2307/1968173.

[Ris70] R. H. Risch. "The Solution of the Problem of Integration in Finite Terms". *Bulletin of the American Mathematical Society* 76.3 (1970), pp. 605–608. DOI: 10.1090/S0002-9904-1970-12454-5.

University of Vienna, Faculty of Mathematics, Oskar-Morgenstern-Platz 1, 1090, Vienna, Austria
Email: florian.fuernsinn@univie.ac.at
Email: herwig.hauser@univie.ac.at

Chubu University, College of Science and Engineering, Matsumoto-cho, Kasugai-shi, Aichi 487-8501, Japan
Kyoto University, Research Institute for Mathematical Sciences, Kitashirakawa-Oiwakecho, Sakyo-ku, Kyoto 606-8502, Japan
Email: kawanoue@kurims.kyoto-u.ac.jp