HOW REGULAR ARE REGULAR SINGULARITIES?

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An abstract look at the work of Fuchs and Frobenius on the solutions of ordinary differential equations at regular singularites.

The differential equation

$$x^2y'' - 3xy' + 3y = 0 \qquad (1)$$

has solutions $y_1 = x$ and $y_2 = x^3$. The equation

$$x^3y''' - 4x^2y'' + 9xy' - 9 = 0 \qquad (2)$$

has likewise the solutions $y_1 = x$ and $y_2 = x^3$, but there seems to hang around one more solution which we missed. Where is it?

An *Euler differential equation* $L_0y = 0$ is given by a linear differential operator of the form

$$L_0 = \sum_{i=0}^{n} c_i x^i \partial^i,$$

with $c_i \in \mathbb{C}$ and $\partial = \partial_x$. It acts on monomials x^ρ via

$$L_0 x^\rho = \chi_0(\rho) \cdot x^\rho,$$

where

$$\chi_0(t) = \sum_{i=0}^{n} c_i t^{\underline{i}}$$

denotes the *indicial polynomial* of L_0. The falling factorial $t^{\underline{i}} = t(t-1)\cdots(t-i+1)$ is called the *Pochhammer symbol*. The above two equations are homogeneous with indicial polynomials $\chi_0(t) = (t-1)(t-3)$ and $\chi_0(t) = (t-1)(t-3)^2$, respectively. This makes it clear that x and x^3 are solutions. But in the second equation, $\rho = 3$ is a double root of χ_0, so something special seems to happen. The classical theory of ordinary linear differential equations tells us that this peculiarity can be understood if we allow logarithms. And, indeed,

$$y_3 = x^3 \log(x)$$

is a third linearly independent solution of the second equation. Up to now, things are straightforward. Let us alter a bit the equations by adding to them a higher degree term, e.g., x^3y. Here, we take the natural grading on $\mathbb{C}[x, \partial]$ given by $\deg(x^i\partial^j) = i - j$ so that Euler operators have degree 0 and x^3y has degree 3. The modification

$$x^2y'' - 3xy' + 3y + x^3y = 0 \qquad (1')$$

of equation (1) has no obvious solutions. It is, however, tempting to expect that its solutions are *perturbations* of x and x^3, say, of the form $x \cdot g(x)$ and $x^3 \cdot h(x)$ for some holomorphic functions g and h not vanishing at 0. Taking an unknown ansatz $g = \sum a_k x^k$ and $h = \sum b_k x^k$ and plugging these power series into the equation yields linear recurrence equations for the coefficients a_k and b_k. They can be solved iteratively.

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Hence g and h exist as formal power series, and then an analysis of the growth behaviour proves their convergence. As analysis is not our main interest here, we skip it.

If we replace the perturbation term x^3y by, say, x^2y, the story makes a brusque turn. The equation

$$x^2y'' - 3xy' + 3y + x^2y = 0 \qquad (1")$$

has a solution of the form $x^3h(x)$, but no longer one of the form $xg(x)$, with g holomorphic and $g(0) \neq 0$.[2] It is not clear at that point whether this is just a coincidence or whether there lies a deeper reason behind the different behaviour of the solutions x and x^3 of equation (1). The situation becomes still more complicated when we perturb the second equation (2). We will return to it later in this text.

Two of the main protagonists in resolving the preceding quandary and in constructing a basis of solutions were Lazarus Immanuel Fuchs (1833–1902, student of Weierstrass) and Ferdinand Georg Frobenius (1849–1917, student of Weierstrass and Kummer) with articles in the Journal für Reine und Angewandte Mathematik in 1866 and 1868, respectively, 1873 [Fu1, Fu2, Fro]. They proposed smart algorithms which produced a precise description of the local solutions of a differential equation near a regular singularity: logarithms, as is now standard knowledge, play a prominent role.

To put their findings on a more conceptual basis, we will rely on a simple though fundamental result from functional analysis about the perturbation of linear operators.

Theorem 1. *Let E be a Banach space and let $L_0, T : E \to F$ be continuous linear operators with values in a topological vector space F. Suppose that the image of T is contained in the image I of L_0 and that the kernel of L_0 admits a closed direct complement H in E,*

$$\mathrm{Ker}(L_0) \oplus H = E.$$

Assume that the inverse $S : I \to H$ of $L_{0|H} : H \to I$ satisfies $|S \circ T| < 1$, where $| - |$ denotes the operator norm induced by the norm of E. Set $L = L_0 + T$. Then $u := \mathrm{Id}_E + S \circ T : E \to E$ is a continuous linear automorphism of E which satisfies

$$L \circ u^{-1} = L_0.$$

Proof. The operator u is well defined since $\mathrm{Im}(T) \subseteq \mathrm{Im}(L_0)$, and continuous because of $|S \circ T| < 1$. Consider the geometric series $v = \sum_{k=0}^{\infty} (-1)^k (S \circ T)^k$. As $|S \circ T| < 1$ and the space of continuous linear operators between Banach spaces is again a Banach space, v defines a continuous linear operator on E. It is then clear that v is inverse to u. Hence u is a continuous linear automorphism of E. We are left to show that $L = L_0 \circ u$. But

$$\begin{aligned} L_0 \circ u &= L_0 \circ (\mathrm{Id}_E + S \circ T) \\ &= L_0 + L_0 \circ S \circ T \\ &= L_0 + L_0 \circ (L_{0|H})^{-1} \circ T \\ &= L_0 + T \\ &= L. \end{aligned}$$

This proves the theorem. ◯

This innocuous looking result is in fact very useful in our context (and in many others). We first apply it to the solution $y_2 = x^3$ of equation (1).

[2] The linear recurrence for the coefficients of $xg(x) = \sum a_k x^{k+1}$ cannot be solved for $a_0 \neq 0$.

Proposition 1. *Let $L_0 = x^2\partial^2 - 3x\partial + 3$ be the Euler operator from above. Let $T = \sum_{ij} c_{ij}x^i\partial^j$ be a second order linear differential operator with polynomial or holomorphic coefficients for which $i - j \geq 1$ whenever $c_{ij} \neq 0$. Set $L = L_0 + T$. Then $Ly = 0$ has a solution of the form $y = x^3h(x)$ with h holomorphic and $h(0) \neq 0$.*

Example. The proposition shows that equations (1') and (1") from above, that is, $x^2y'' - 3xy' + 3y + x^3y = 0$ and $x^2y'' - 3xy' + 3y + x^2y = 0$, have both a solution of the form $y = x^3h(x)$ with h holomorphic and $h(0) \neq 0$.

Proof. We first look for formal power series solutions of $Ly = 0$. The operator L_0 sends $x^3\mathbb{C}[[x]]$ onto $x^4\mathbb{C}[[x]]$ and its restriction to $x^4\mathbb{C}[[x]]$ (which is a direct complement of the kernel $x^3\mathbb{C}$ of L_0) gives an automorphism of $x^4\mathbb{C}[[x]]$. Let S be its inverse. As $x^4\mathbb{C}[[x]]$ contains $T(x^3\mathbb{C}[[x]])$ by the assumption on T the composition $S \circ T : x^3\mathbb{C}[[x]] \to x^4\mathbb{C}[[x]] \subseteq x^3\mathbb{C}[[x]]$ is well defined. It is easy to see that applying the operator $S \circ T$ increases the order of power series in $x^3\mathbb{C}[[x]]$. Therefore $v = \sum_{k=0}^{\infty}(-1)^k(S \circ T)^k$ defines an automorphism of $x^3\mathbb{C}[[x]]$ (use here that $\mathbb{C}[[x]]$ is complete with respect to the x-adic topology). It is inverse to $u = \mathrm{Id}_{x^3\mathbb{C}[[x]]} + S \circ T$. The same calculation as in the proof of Theorem 1 shows that $L \circ v = L \circ u^{-1} = L_0$ holds on $x^3\mathbb{C}[[x]]$. We conclude that $y = u^{-1}(x^3) =: x^3h(x) \in x^3\mathbb{C}[[x]]$ is a formal solution of $Ly = 0$ with $h(0) \neq 0$.

Convergence is more delicate: For a power series $h(x) = \sum_{k=0}^{\infty} a_kx^k$ and $s > 0$ we set $|h|_s = \sum_{k=0}^{\infty} |a_k|s^k$ and let $\mathbb{C}\{x\}_s$ denote the Banach space of convergent series h with finite s-norm $|h|_s < \infty$. Note that $\mathbb{C}\{x\}_s \subseteq \mathbb{C}\{x\}_{s'}$ for $0 < s' \leq s$ and $\mathbb{C}\{x\} = \bigcup_{s>0} \mathbb{C}\{x\}_s$. It is well known that differentiation $h \to h'$ sends $\mathbb{C}\{x\}_s$ into $\mathbb{C}\{x\}_{s'}$ for all $0 < s' < s$, see [GrRe, Satz 3, p. 19], but not necessarily into $\mathbb{C}\{x\}_s$, just take $h = \sum_{k=1}^{\infty} \frac{x^{k+1}}{k(k+1)} \in \mathbb{C}\{x\}_1$ with $h' \notin \mathbb{C}\{x\}_1$.

Let $E_s = x^3\mathbb{C}\{x\}_s$ for $s > 0$, $F = \mathbb{C}\{x\}$, and write $L_0, T : E_s \to F$ also for the restrictions of L_0 and T to E_s. From the first part of the proof we already dispose of the map $S \circ T : x^3\mathbb{C}[[x]] \to x^3\mathbb{C}[[x]]$ between formal power series spaces. The crucial point now is to show that $S \circ T$ sends $E_s = x^3\mathbb{C}\{x\}_s$ to itself and that the restriction $(S \circ T)_s : E_s \to E_s$ satisfies $|(S \circ T)_s| < 1$ for $s > 0$ sufficiently small. But a convergent $h = \sum_{k=0}^{\infty} a_kx^k$ is sent by $S \circ T$ to

$$(S \circ T)(h) = S\left(\sum_{ij} c_{ij} \sum_k a_k k^{\underline{j}} x^{k+i-j}\right) = \sum_{ij} \sum_k \frac{k^{\underline{j}}}{(k+i-j-1)(k+i-j-3)} c_{ij}a_kx^{k+i-j},$$

by the very definitions of T and S. Recall here that $\chi_{L_0}(t) = (t-1)(t-3)$ is the indicial polynomial of L_0 and that S is the inverse of $L_{0|x^4\mathbb{C}\{x\}}$. As $j \leq 2$, the ratios $\frac{k^{\underline{j}}}{(k+i-j-1)(k+i-j-3)}$ remain bounded as i and k go to ∞. Therefore $(S \circ T)(h)$ converges again. But as $i - j \geq 1$ for all i, j with $c_{ij} \neq 0$, a short computation shows that one can even achieve

$$|(S \circ T)_s(h)|_s \leq C|h|_s$$

for all $0 < s < s_0$ and some constant $C < 1$ independent of h, provided that s_0 is sufficiently small (s_0 will also be independent of h). Therefore $|(S \circ T)_s| < 1$. This shows that the restriction $u_s : E_s \to E_s$ of u is a well defined continuous linear automorphism with inverse the restriction v_s of v.

By Theorem 1, we now get $L \circ u_s^{-1} = (L_0 + T) \circ u_s^{-1} = L_0$ on E_s. This implies that $y = u_s^{-1}(x^3) = x^3h(x)$ with h holomorphic and $h(0) \neq 0$ is a solution of $Ly = 0$. ○

Remark. The argument applied to x^3 in the preceding examples (1') and (1") does not work equally well for the solution $y_1 = x$ of equation (1), even when looking just for formal solutions: We would have to let L_0 operate on the space $x\mathbb{C}[[x]]$, observing that the image of L_0 now equals $x^2\mathbb{C} \oplus x^4\mathbb{C}[[x]]$, with a *gap* at x^3. For equation (1'), the image of the operator T defined by $Ty = x^3 y$ is $x^4\mathbb{C}[[x]]$. It is contained in the image of L_0 and one can find the solution $xg(x)$ of equation (1') satisfying $g(0) \neq 0$ as in the proposition. On the other hand, for equation (1"), the image of the operator T defined by $Ty = x^2 y$ equals $x^3\mathbb{C}[[x]]$ and is thus *not* contained in the image of L_0. The reasoning does not apply.

> Lazarus Fuchs and Georg Frobenius would now potentially say the following: The roots of the indicial polynomial of (1) are 1 and 3, the *local exponents* of equation (1) at 0. They are congruent modulo $\mathbb{Z}$. The larger one, 3, is maximal: $3 + k$ is not a root for any positive integer k. Therefore, the local solutions of (1') and (1") at 0 with respect to the maximal exponent $\rho = 3$ are of the form $x^3 h(x)$ with h holomorphic, regardless of the perturbation term.
>
> And F. G. F. would continue: The exponent $\sigma = 1$, however, is not maximal. Extra caution has to be taken to find the solutions $xg(x)$ with $g(0) \neq 0$. A nice trick would be to consider a formal expression $x^t h(x)$, where now t is a variable, and to try to solve the equation for this term. One will fail, but may observe that differentiating the whole situation with respect to t does produce a solution, now involving a logarithm (it comes from the differentiation with respect to the exponent). This mysterious manipulation is suggested and explained in my paper [Fro]. The prospective solution of equation (1") corresponding to the exponent $\sigma = 1$ will be of the form $y_1 = x[g(x) + h(x)\log(x)]$. Perfect!

Let $L \in \mathbb{C}\{x\}[\partial]$ be a linear differential operator with polynomial or holomorphic coefficients, $L = \sum c_{ij} x^i \partial^j$. Consider the vector space $\mathbb{C}\{x\}[z]$ of polynomials in a new variable z with coefficients in $\mathbb{C}\{x\}$. The *extension* $\mathbb{L}$ of L to $\mathbb{C}\{x\}[z]$ is defined as

$$\mathbb{L} = \sum c_{ij} x^i \boldsymbol{\partial}^j,$$

where the derivation $\boldsymbol{\partial}$ on $\mathbb{C}\{x\}[z]$ is prescribed by the two requirements $\boldsymbol{\partial} x = 1$ and $\boldsymbol{\partial} z = x^{-1}$ (compare this with section 4 in [Hon]).

Lemma 1. *If L_0 is an n-th order Euler operator with indicial polynomial χ_0 then*

$$\mathbb{L}_0(x^t z^k) = x^t \cdot [\chi_0(t) z^k + \chi_0'(t) k z^{k-1} + \ldots + \tfrac{1}{n!}\chi_0^{(n)}(t) k^{\underline{n}} z^{k-n}].$$

Proof. Pastime for the dentist's waiting room.[3] $\circlearrowleft$

Lemma 2. *If L_0 is an Euler operator and $\rho \in \mathbb{C}$ is an m-fold root of χ_0, then*

$$x^\rho,\ x^\rho z,\ \ldots,\ x^\rho z^{m-1}$$

are solutions of $\mathbb{L}_0 y = 0$ in $\mathbb{C}\{x\}[z]$. $\circlearrowleft$

Define, for $\ell \geq 0$, linear maps $(\partial^j)^{(\ell)} : \mathbb{C}\{x\} \to \mathbb{C}\{x\}$ by

$$(\partial^j)^{(\ell)}(x^t) = (t^{\underline{j}})^{(\ell)} x^{t-j}.$$

Note that $(\partial^j)^{(0)} = \partial^j$ is the usual j-fold differentiation. For an arbitrary differential operator $L = \sum_{j=0}^n \sum_{i=0}^\infty c_{ij} x^i \partial^j$ we call

$$L^{(\ell)} = \sum_{j=0}^n \sum_{i=0}^\infty c_{ij} x^i (\partial^j)^{(\ell)}$$

[3] It suffices to prove this for $\mathbb{L}_0 = \boldsymbol{\partial}^j$. One may replace z by $\log(x)$ and apply the usual differentiation rules, replacing afterwards all powers $\log(x)^i$ by z^i again.

the ℓ-*th Pochhammer derivative* of L. This is no longer a differential operator in the classical sense, but, of course, defines again a linear map $L^{(\ell)} : \mathbb{C}\{x\} \to \mathbb{C}\{x\}$. If $L = L_0$ is an Euler operator with indicial polynomial χ_0, then

$$L_0^{(\ell)}(x^t) = \chi_0^{(\ell)}(t)x^t,$$

where $\chi_0^{(\ell)}$ denotes the ℓ-fold derivative of χ_0. This fact motivates the differentiation notation $(\partial^j)^{(\ell)}$. One may call $\chi_0^{(\ell)}$ the indicial polynomial of $L_0^{(\ell)}$.

Leo August Pochhammer (1841–1920, student of Kummer) may complain here that his name is used without having been asked. He may also request, possibly in latin, to find an axiomatic characterization of the linear maps $(\partial^j)^{(\ell)} : \mathbb{C}\{x\} \to \mathbb{C}\{x\}$.

Lemma 3. *The extension $\mathbb{L}$ to $\mathbb{C}\{x\}[z]$ of an n-th order differential operator L has "Taylor expansion"*

$$\mathbb{L} = L + L'\partial_z + \tfrac{1}{2}L''\partial_z^2 + \ldots + \tfrac{1}{\ell!}L^{(\ell)}\partial_z^\ell + \ldots + \tfrac{1}{n!}L^{(n)}\partial_z^n,$$

where now L, L', ... are taken as maps on $\mathbb{C}\{x\}[z]$, acting on the coefficients in $\mathbb{C}\{x\}$ of the polynomials in z while leaving z invariant, and where ∂_z denotes the usual differentiation with respect to z.

Proof. As for Lemma 1. ○

Lemma 4. $\mathbb{L}(x^i z^k)_{|z=\log(x)} = L(x^i \log(x)^k)$. ○

Let $L = \sum_{j=0}^n \sum_{i=0}^\infty c_{ij}x^i\partial^j$ be an n-th order differential operator in $\mathbb{C}\{x\}[\partial]$. Its *initial form* is the homogeneous operator $L_0 = \sum_{i-j=\tau} c_{ij}x^i\partial^j$ where the *shift* τ is the minimal difference $i - j$ for which some c_{ij} is non zero. Up to multiplication of L by a Laurent monomial we may and will assume that the shift τ is 0, i.e., that L_0 is an Euler operator. The origin 0 is called a *regular singularity* of L if L_0 is again an operator of order n. It just means that $c_{nn} \neq 0$. This is an important concept. It ensures that formal power series solutions of $Ly = 0$ are actually convergent. Many characterizations exist [Sin, Proposition 1.4.2, vdPS, section 5.1]. The operator L of Proposition 1 has a regular singularity at 0 and initial form L_0.

The indicial polynomial χ_L of an operator L is defined as the indicial polynomial χ_0 of L_0, and its roots $\rho \in \mathbb{C}$ are the *local exponents* of L at 0. They may be multiple roots as in equation (2).

Proposition 2. *Let L be an operator with regular singularity at 0 and initial form $L_0 = x^3\partial^3 - 4x^2\partial^2 + 9x\partial - 9$ as in equation (2) above. Then $Ly = 0$ has two solutions of the form $y_1 = x^3h(x)$ and $y_2 = x^3[g(x) + h(x)\log(x)]$ with g, h holomorphic and $g(0), h(0) \neq 0$.*

Proof. Write $L = L_0 + T$ with T an operator of shift ≥ 1 and denote by $\mathbb{L}_0$ and $\mathbb{T}$ the extensions of L_0 and T to the subspace

$$E = x^\rho\mathbb{C}\{x\} \oplus x^\rho\mathbb{C}\{x\}z = x^3\mathbb{C}\{x\} \oplus x^3\mathbb{C}\{x\}z$$

of $x^3\mathbb{C}\{x\}[z]$. The image of $\mathbb{L}_0$ equals $x^4\mathbb{C}\{x\} \oplus x^4\mathbb{C}\{x\}z$ since $\rho = 3$ is a maximal exponent of L_0. It thus contains the image of $\mathbb{T}$. Restricting to power series in x of finite s-norm the theorem applies. An appropriate operator $\mathbb{S}$ depends on the choice of a direct complement H of the kernel of $\mathbb{L}_0$ in E. One preferably takes here $H = x^{\rho+1}\mathbb{C}\{x\} \oplus x^{\rho+1}\mathbb{C}\{x\}z = x^4\mathbb{C}\{x\} \oplus x^4\mathbb{C}\{x\}z$. To establish the norm estimate $|\mathbb{S} \circ \mathbb{T}| < 1$ for the restriction to suitable Banach spaces E_s, for $s > 0$ sufficiently small, a similar estimate as in the proof of Proposition 1 works, using now that 0 is a regular singularity of L.[4] One gets $\mathbb{L} \circ u^{-1} = \mathbb{L}_0$. Now pull back with u^{-1} the solutions x^3 and $x^3 z$ of $\mathbb{L}_0 y = 0$ to get the solutions of $\mathbb{L} y = 0$ in E. Then replace there z by $\log(x)$. The resulting solutions y_1 and y_2 of $Ly = 0$ are of the form as indicated.[5] ⟲

At that point, the attentive reader may have observed that the second exponent $\sigma = 1$ and the solution $y_1 = x$ of the Euler equation $L_0 y = 0$ have been neglected in the statement of Proposition 2. There is a reason for this:

Let L_0 have two local exponents ρ and σ which are congruent to each other modulo $\mathbb{Z}$, that is, $\sigma = \rho - e$, $e \in \mathbb{N}_{>0}$. Consider the subspace

$$E = x^\sigma \mathbb{C}\{x\} \oplus x^\rho \mathbb{C}\{x\} z$$

of $x^\sigma \mathbb{C}\{x\}[z]$ and the extension

$$\mathbb{L}_0 : E \to E$$

of L_0 to it. What is its image?

Let $L_0 = x^2 \partial^2 - 3x\partial + 3$ be the operator from equation (1). We have $\rho = 3$ and $\sigma = 1$. A gentle computation gives (for general ρ and σ)

$$\mathbb{L}_0(x^\sigma + x^\rho z) = [\chi_0(\sigma)x^\sigma + \chi_0'(\rho)x^\rho] + [\chi_0(\rho)x^\rho]z,$$

say,

$$\mathbb{L}_0(x^\sigma + x^\rho z) = [(\sigma-1)(\sigma-3)x^\sigma + 2(\rho-2)x^\rho] + [(\rho-1)(\rho-3)x^\rho]z.$$

A short moment of reflection will confirm that the image $\mathbb{L}_0(E)$ of $\mathbb{L}_0$ is therefore

$$x^{\sigma+1}\mathbb{C}\{x\} \oplus x^{\rho+1}\mathbb{C}\{x\}z = x^2\mathbb{C}\{x\} \oplus x^4\mathbb{C}\{x\}z.$$

Both summands have *no* gaps. Sounds good!

Proposition 3. *Let L be differential operator with regular singularity at 0 and initial form $L_0 = x^2\partial^2 - 3x\partial + 3$ as in equation (1) above. Then $Ly = 0$ has a basis of local solutions of the form $y_1 = x^3 f(x)$ and $y_2 = xg(x) + x^3 h(x)\log(x)$ with f, g and h holomorphic and none vanishing at 0.*

Proof. Write $L = L_0 + T$. Denote by $\mathbb{T}$ the extension of T to $x^\sigma\mathbb{C}\{x\} \oplus x^\rho\mathbb{C}\{x\}z$. Its image is contained in the image of $\mathbb{L}$ as computed before, because T has shift ≥ 1. Restricting to power series of finite s-norm, for $s > 0$ sufficiently small, the theorem applies to $\mathbb{L}$. Lift with u^{-1} the solutions x and x^3 of $\mathbb{L}_0 y = 0$ to solutions of $\mathbb{L} y = 0$. Then replace z by $\log(x)$. ⟲

Lothar Heffter (1862–1962, student of Fuchs) would now possibly reply that this perturbation method already appeared in his book from 1894 [Hef]. This could to a certain extent be correct, though it is hard to compare the two approaches in detail. The reader may consult [Poo, section V.16] or [Mez, section 4.4] to get a personal opinion.

We can now understand why for equation (1')

$$x^2 y'' - 3xy' + 3y + x^3 y = 0$$

[4] The critical ratio in the expansion of $(\mathbb{S} \circ \mathbb{T})(h)$ is now $\frac{k^j}{\chi_{L_0}(k+i-j)}$. The assumption on the regularity of the singularity signifies that the indicial polynomial χ_{L_0} has degree equal to the order of L, hence the ratio is bounded from above as k and i tend to ∞.

[5] The reappearence of the function h in y_2 is due to the special shape of u; the proof would require some computation.

it is easier to construct a basis of solutions than for (1")

$$x^2y'' - 3xy' + 3y + x^2y = 0.$$

In the first case, the perturbation term T is defined by $Ty = x^3y$. Its restriction to $x^\sigma\mathbb{C}\{x\} = x\mathbb{C}\{x\}$ has image $x^4\mathbb{C}\{x\}$ contained in the image of $L_{|x\mathbb{C}\{x\}}$ which is $\mathbb{C}x^2 \oplus x^4\mathbb{C}\{x\}$. This does not happen for equation (1"), where T is defined by $Ty = x^2y$ and has image $x^3\mathbb{C}\{x\}$ not contained in the image of L_0. In the first case, we find an isomorphism u of $x\mathbb{C}\{x\}$ such that $L \circ u^{-1} = L_0$, whereas in the second case we have to resort to $E = x\mathbb{C}\{x\} \oplus x^3\mathbb{C}\{x\}z$ and the extensions $\mathbb{L}$ and $\mathbb{T}_0$ of L and T to normalize $\mathbb{L}$ to $\mathbb{L}_0$ by an automorphism of E. This creates the presence of logarithms in the respective solutions.

One may want to try to work instead with the extension $\mathbb{L}$ induced by L on the larger space $\widetilde{E} = x^\sigma\mathbb{C}\{x\} \oplus x^\sigma\mathbb{C}\{x\}z = x\mathbb{C}\{x\} \oplus x\mathbb{C}\{x\}z$ instead of $E = x\mathbb{C}\{x\} \oplus x^3\mathbb{C}\{x\}z$. This would be suggested by the description of solutions by Frobenius [Fro, p. 222]. However, one can check that for the extension of $\mathbb{L}$ to $\widetilde{E}$ the normalization procedure of the theorem does not go through.

In Proposition 1 we constructed one solution for one simple maximal exponent, in Proposition 2 two solutions for one maximal exponent of multiplicity 2, in Proposition 3 two solutions for one simple maximal and one non maximal exponent. There is one more case: one maximal exponent of multiplicity 2 and one non maximal exponent.

Proposition 4. *Let L be differential operator with regular singularity at 0 and initial form $L_0 = x^3\partial^3 - 4x^2\partial^2 + 9x\partial - 9$ as in equation (2) above. Then $Ly = 0$ has a basis of local solutions at 0 of the form*

$$y_1 = x^3 f_0(x),$$

$$y_2 = x^3[f_1(x) + f_0(x)\log(x)],$$

$$y_3 = xg(x) + x^3h_1(x)\log(x) + x^3h_2(x)\log(x)^2,$$

with f_0, f_1, g, h_1, and h_2 holomorphic and none vanishing at 0.

Proof. For the first two solutions we may work with the extensions $\mathbb{L}$ and $\mathbb{T}$ of L and T to $E = x^3\mathbb{C}\{x\} \oplus x^3\mathbb{C}\{x\}z$ and apply the same arguments as in the proof of Proposition 2. For the third solution, things are getting more complicated, since we now have to extend L and T to the space

$$E' = x\mathbb{C}\{x\} \oplus x^3\mathbb{C}\{x\}z \oplus x^3\mathbb{C}\{x\}z^2.$$

Call $\mathbb{L}$ and $\mathbb{T}$ the respective extensions. Again, the composition $\mathbb{L} \circ u^{-1} = \mathbb{L}_0$ normalizes $\mathbb{L}$ on E' to its initial form $\mathbb{L}_0$ for a suitable automorphism u of E'. To obtain the exact shape of the third solution, one has to choose $\mathbb{S}$ suitably as in the proof of Proposition 2 and then look carefully at the action of u on the three summands of E. This is a bit tedious and will be omitted.[6] ⟲

At least now it should have become clear how the story continues: the local exponents of a given differential operator L with regular singularity at 0 have to be listed in sets of exponents congruent to each other modulo $\mathbb{Z}$. For the largest elements of each of these sets, say ρ, with multiplicity m as a root of χ_L, arguments as used for Propositions 1 and 2 apply, yielding solutions

$$y_k(x) = x^\rho\left[f_{k-1}(x) + f_{k-2}(x)\log(x) + \ldots + f_0(x)\log(x)^{k-1}\right],$$

[6] A comprehensive manual for this type of extension techniques is in preparation [Hau].

for $k = 1, ..., m$ and with holomorphic functions f_0, ..., f_{m-1}, none vanishing at 0. The second largest exponent of the set containing ρ, say σ, of multiplicity ℓ, yields solutions of the form

$$y_k(x) = x^\sigma \left[g_{k-1}(x) + g_{k-2}(x)\log(x) + \ldots + g_0(x)\log(x)^{k-1}\right] + \\ + x^\rho \log(x)^k \left[h_{m-1}(x) + h_{m-2}(x)\log(x) + \ldots + h_0(x)\log(x)^{m-1}\right],$$

for $k = 1, ..., \ell$ and with holomorphic g_0, ..., $g_{\ell-1}$, h_0, ..., h_{m-1}, none vanishing at 0.[7] And so on for the other exponents.[8]

Theorem 2. *Every n-th order linear differential operator L with regular singularity at 0, initial form L_0, and local exponents $\rho \in \Omega$, admits an extension $\mathbb{L}$ to a finite rank $\mathbb{C}\{x\}$-submodule E of $\prod_{\rho\in\Omega} x^\rho\mathbb{C}\{x\}[z]$ which contains a basis of solutions of $\mathbb{L}_0$ at 0 and such that*

$$\mathbb{L} \circ u^{-1} = \mathbb{L}_0,$$

where the normalizing automorphism $u = \mathrm{Id}_E + \mathbb{S} \circ \mathbb{T}$ of E is built from an "inverse" $\mathbb{S}$ of $\mathbb{L}_0$ and the higher degree terms $\mathbb{T} = \mathbb{L} - \mathbb{L}_0$ of $\mathbb{L}$. A basis of local solutions of $Ly = 0$ can be computed by pulling back via u^{-1} the trivial solutions of $\mathbb{L}_0$ in E and setting $z = \log(x)$.

> Frobenius: So it seems that regular singularities are even more regular than my dear colleague, Herr Fuchs, may have suspected. Fuchs: Yes, lieber Herr Frobenius, this, indeed, seems to be the case.

Idea of proof. The $\mathbb{C}\{x\}$-module E will be a cartesian product $E = \prod_{\rho\in\Omega} E_\rho$ of finite rank $\mathbb{C}\{x\}$-modules E_ρ attached to each $\rho \in \Omega$. We sketch their construction: Let Ω_1 be a set of local exponents all of which are congruent to each other modulo $\mathbb{Z}$ and such that no other exponent in Ω is congruent modulo $\mathbb{Z}$ to an element of Ω_1. This set is naturally ordered by considering differences. Let ρ be the maximal exponent in Ω_1, with multiplicity m, and let σ be the second largest exponent in Ω_1, with multiplicity ℓ. The factor E_ρ of E taking care of ρ is of the form

$$E_\rho = x^\rho\mathbb{C}\{x\} \oplus x^\rho\mathbb{C}\{x\}z \oplus \cdots \oplus x^\rho\mathbb{C}\{x\}z^{m-1}.$$

The summand E_σ of E taking care of σ is of the form

$$E_\sigma = x^\sigma\mathbb{C}\{x\} \oplus \cdots \oplus x^\sigma\mathbb{C}\{x\}z^{\ell-1} \oplus x^\rho\mathbb{C}\{x\}z^\ell \oplus \cdots \oplus x^\rho\mathbb{C}\{x\}z^{\ell+m-1}.$$

Compare these formulas with the description of the solutions y_1, y_2 and y_3 in Proposition 4. It can now be guessed how the construction of E continues for smaller exponents in Ω_1. To establish the theorem, it then suffices to show that the image of $\mathbb{L}_0$ on each E_ρ has no gaps, with the exponents of x^ρ, x^σ, ..., shifted by 1. To illustrate, one has to show that

$$\mathbb{L}_0(E_\rho) = x^{\rho+1}\mathbb{C}\{x\} \oplus \cdots \oplus x^{\rho+1}\mathbb{C}\{x\}z^{m-1},$$

$$\mathbb{L}_0(E_\sigma) = x^{\sigma+1}\mathbb{C}\{x\} \oplus \cdots \oplus x^{\sigma+1}\mathbb{C}\{x\}z^{\ell-1} \oplus x^{\rho+1}\mathbb{C}\{x\}z^\ell \oplus \cdots \oplus x^{\rho+1}\mathbb{C}\{x\}z^{\ell+m-1}.$$

Proving these equalities is a purely combinatorial task, using the formula from Lemma 1. They imply that the image of each E_ρ under $\mathbb{T}$ is contained in the image of $\mathbb{L}_0$. This is required in order to be able to apply Theorem 1 and to get $\mathbb{L} \circ u^{-1} = \mathbb{L}_0$. ⥀

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[7] Each y_k has $k + m$ summands, for $k = 1, ..., \ell$, but there appear, in total, only $\ell + m$ different coefficient functions.

[8] The respective formulas (11) in [Fu1, p. 136], (12) in [Fro, p. 222], and in [Ince, p. 401] are somewhat cumbersome to disentangle.

Remarks. (a) In the case that L has polynomial coefficients, $L \in \mathbb{C}[x, \partial]$, the automorphism u restricts to a linear endomorphism u° of $E^\circ = E \cap \sum_{\rho \in \Omega} x^\rho \mathbb{C}[x][z]$ which may be called *differentially étale*: its degree 0 term is the identity, whereas the remaining terms increase the order in x at 0; after extending E° to an analogous $\mathbb{C}[[x]]$-submodule $\widehat{E}$ of $\prod_{\rho \in \Omega} x^\rho \mathbb{C}[[x]][z]$ by allowing formal power series, the extension of u° to $\widehat{E}$ becomes an automorphism $\widehat{u} : \widehat{E} \to \widehat{E}$. It is a mystery which formal power series appear in the coefficients of z^i in the images under $\widehat{u}^{-1}$ of polynomials in E°. They must be quite special![9]

(b) The hypothesis in the theorem that L has a regular singularity at 0 is used to ensure the norm estimate required for the application of Theorem 1. This estimate was responsible to have u^{-1} map convergent series to convergent ones. Dropping the regularity condition, one still gets a normalizing automorphism u but now for formal power series with a $\mathbb{C}[[x]]$-submodule $\widehat{E}$ of $\prod_{\rho \in \Omega} x^\rho \mathbb{C}[[x]][z]$. The convergence of the geometric series to an operator defining u^{-1} is deduced in this case from the fact that free finite rank $\mathbb{C}[[x]]$-modules are complete with respect to the x-adic topology (as it was used with $\mathbb{C}[[x]]$ itself at the beginning of the proof of Proposition 1). With respect to the solutions of the equations, they have precisely the same shape as in the regular singular case, but the "coefficient functions" f_i, g_i and h_i are now just formal power series and need no longer be holomorphic.

One has to be cautious here as the order of L_0 is strictly smaller than the order of L if the singularity is irregular. Hence one does not get n linearly independent formal solutions but just as many as the order of L_0 indicates.[10]

(c) For an extension of Thm. 2 to several variables, but restricting to formal power series (no logarithms and powers with complex exponent), see the Monomialization Theorem in [GH, p. 11]. In this paper, a generalization of the concept of regular singularity for partial differential equations is proposed.

(d) It is tempting to have a glance at differential equations with polynomial or formal power series coefficients defined over a field of positive characteristic $p > 0$. Logarithms no longer exist, and already $y' = y$ has no solution. Does there exist a (significant) variant of Theorem 2 which is valid in positive characteristic?

One more pastime. Every linear differential equation $Ly = 0$ with holomorphic coefficients at 0 is equivalent to a system of first order linear differential equations, $Y' = AY$, where $Y = (y_1, ..., y_n)^t$ is a column vector of unknown functions and $A \in \mathrm{M}_n(\mathrm{Quot}(\mathbb{C}\{x\}))$ is a matrix with meromorphic entries. Formulate and prove the normal form theorem from above in terms of such systems and the matrix A (you may compare your findings with Thm. 1 in [Turr], see also [Lev]).

How regular are regular singularities? Even though we have not given all details (actually, we gave only very few), we hope to have conveyed the flavour of the techniques: The statement in Theorem 2 is a normal form theorem for differential operators acting on specific spaces of formal or convergent power series (namely, on the spaces $\widehat{E}_\rho$ and E_ρ). It asserts that all relevant information is contained, up to an automorphism of the source space, in the initial form of the operator. This holds in both the formal and the convergent setting, in the first case regardless of whether the singularity is regular or not. But if it is regular, one gets in addition that the normalizing automorphism sends convergent series to convergent series. Therefore, differential equations with regular singularities are locally equivalent to the Euler equation given by the initial form of the operator. As such, regular singularities behave also regularly in this sense.

[9] This comment may seem itself to be a bit mysterious. To catch its perspective requires a moment of contemplation.

[10] We are indebted to a referee for emphasizing this detail.

The normal form of the operator has an immediate consequence on the description of the solutions of the differential equation defined by the operator: One solves the initial equation and lifts its solutions with the normalizing automorphism to the solutions of the original equation. This, again, works for the formal and convergent setting. And as a by-product, similar perturbation methods apply to prove more generally the Malgrange index theorem [Mal, Kom]: It counts the number of convergent solutions of irregular singular differential equations among all formal solutions. For the case of several variables, we refer to the paper [GH] where the notion of perfect operator gives a prospective generalization of regular singularities to partial differential equations.

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Bibliography

The article [Fro] of Frobenius is truly inspiring and cristal-clear, though compact. It takes up and extends ideas and constructions of Fuchs [Fu1, Fu2], papers which are a bit more outspread but layed the basis of the theory. The proof that one obtains indeed a basis of solutions – this is only sketched in [Fro] – is detailed in [Tho] and [Ince, p. 402]. These methods are reproduced, among many other sources and in slightly different language, in [Poo, Mez, Tou, CH, Sau]. The book [Poo] of Poole and the thesis [Mez] describe also the recursive approach of Heffter [Hef]. For more on regular singularities, see [vdPS], [Sin], [Har] and [Was]. Historical information can be found in [Gra, Schl]. Finally, the article [GH] extends the methods of the present text to the multivariate case of partial differential equations.

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