FUCHSIAN DIFFERENTIAL EQUATIONS: NOTES FALL 2022

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NOTES PART VI: NORMAL FORM THEOREM, GENERAL CASE

Up to now we have always considered a local exponent ρ of $L \in \mathcal{O}[6]$ which was maximal modulo $\mathbb{Z}$, and we proved for this case the normal form theorem on the function space $\mathcal{F} = x^\rho \mathcal{O}[z]_{<m}$, where m is the multiplicity of ρ as a root of the indicial polynomial χ of L. We will now treat the general case of arbitrary local exponents. It turns out that local exponents which differ by an integer pose extra problems. We will call the occurrence of integer differences $resonance$.

To motivate our constructions, let us recall that the key step in the proof of the normal form theorem was to determine the image $\underline{L}_0(\mathcal{F})$ of the initial form L_0 of L and to prove that actually $L_0(\mathcal{F}) = x\mathcal{F}$. From this follows that the tail $T = L_0 - L$ of L sends $\mathcal{F}$ into the image of L_0, which is the critical property used to construct the normalizing automorphism u.

Let us illustrate first that a suitable definition of $\mathcal{F}$ is so obvious in case of resonance.

$Example.$ Let $L_0 = x^2\partial^2 - x\partial$ be an Euler operator of shift 0 and with indicial polynomial $\chi(t) = t(t-2)$. The local exponents are $\sigma = 0$ and $\rho = 2$, both of multiplicity 1. As both σ and ρ are simple roots of χ, one may expect that we can dispense of logarithms. A natural candidate for $\mathcal{F}$ seems to be $\mathcal{F} = x^\sigma\mathcal{O} + x^\rho\mathcal{O} = \mathcal{O} + x^2\mathcal{O} = \mathcal{O}$. Let us compute its image under L_0. We get

$$L_0(\mathcal{O}) = \mathbb{C}x + x^3\mathcal{O} \subsetneq x\mathcal{F} = x\mathcal{O}.$$

So the image is strictly contained in $x\mathcal{F}$. If you now take $L = L_0 - x^2\partial$ with $T = x^2\partial$, we see that $T(x) = x^2$ is not contained in $x\mathcal{F}$. So the construction of the automorphism u breaks down.

To remedy this failure, let us introduce logarithms in $\mathcal{F}$. We will describe two options to do this. The first one turns out to be unsuccessful, while the second will work.

$Attempt$ 1. Take

$$\mathcal{F} = (x^\sigma\mathcal{O} + x^\rho\mathcal{O})[z]_{<2} = \mathcal{O}[z]_{<2} = \mathcal{O} \oplus \mathcal{O}z.$$

This looks like a reasonable choice. Let us write $\underline{L}_0 = x^2\underline{\partial}^2 - x\underline{\partial}$ for the action of the extension of L_0 to $\mathcal{F}$. We get

$$\underline{L}_0(\mathcal{F}) = \mathbb{C}x + x^3\mathcal{O} + \underline{L}_0(\mathcal{O}z).$$

To compute the last summand, recall from Lemma 3 in part III that

$$\underline{L}_0(x^k z) = x^k[\chi(k)z + \chi'(k)].$$

As $\chi(t) = t(t-2)$ and $\chi'(t) = 2(t-1)$ this gives

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$k = 0$: $\quad \underline{L}_0(z) = 0z + 2(0-1) = -2;$

$k = 1$: $\quad \underline{L}_0(xz) = -xz + 0 = -xz;$

$k = 2$: $\quad \underline{L}_0(x^2z) = 0x^2z + 2x^2 = 2x^2;$

$k = 3$: $\quad \underline{L}_0(x^3z) = 3x^3z + 4x^3.$

Already the first case $k = 0$ shows that $\underline{L}_0(\mathcal{F}) \not\subset x\mathcal{F}$. So this choice of $\mathcal{F}$ is not appropriate.

Attempt 2. Let us now take

$$\mathcal{F} = x^\sigma \mathcal{O}[z]_{<1} + x^\rho \mathcal{O}[z]_{<2} = \mathcal{O} + x^2\mathcal{O} + x^2\mathcal{O}z = \mathcal{O} \oplus x^2\mathcal{O}z.$$

We have $\underline{L}_0(\mathcal{F}) = \underline{L}_0(\mathcal{O}) + \underline{L}_0(x^2\mathcal{O}z)$ with $\underline{L}_0(\mathcal{O}) = L_0(\mathcal{O}) = \mathbb{C}x + x^3\mathcal{O}$ as before. As for $\underline{L}_0(x^2\mathcal{O}z)$, use again Lemma 3 to compute $\underline{L}_0(x^kz)$. We get as before

$k = 2$: $\quad \underline{L}_0(x^2z) = 2x^2;$

$k = 3$: $\quad \underline{L}_0(x^3z) = 3x^3z + 4x^3.$

$k = 4$: $\quad \underline{L}_0(x^3z) = 8x^4z + 6x^4.$

This implies

$$\underline{L}_0(\mathcal{F}) = \underline{L}_0(\mathcal{O} \oplus x^3\mathcal{O}z) = \mathbb{C}x + x^3\mathcal{O} + \mathbb{C}x^2 + x^3\mathcal{O}z = x(\mathcal{O} \oplus x^2\mathcal{O}z) = x\mathcal{F}.$$

That is precisely what we want - and it gives us a hint of how to define $\mathcal{F}$ in general.

Lemma 5. *Let $E \in \mathcal{O}[z]$ be an Euler operator with shift 0. Let Ω be a set of local exponents of E with integer differences, ordered increasingly,*

$$\rho_1 < \rho_2 < \ldots < \rho_r,$$

meaning that $\rho_{k+1} - \rho_k \in \mathbb{N}_{>0}$. Let m_k be the multiplicity of ρ_k. Set

$$\mathcal{F} = x^{\rho_1}\mathcal{O}[z]_{<m_1} + x^{\rho_2}\mathcal{O}[z]_{<m_1+m_2} + \ldots + x^{\rho_r}\mathcal{O}[z]_{<m_1+m_2+\ldots+m_r}.$$

Then

$$\underline{E}(\mathcal{F}) = x\mathcal{F}.$$

Proof. (a) We show that $\underline{E}(\mathcal{F}) \subset x\mathcal{F}$. Recall from Lemma 3 that

$$\underline{E}(x^\rho z^i) = x^\rho \cdot [\chi(\rho)z^i + \chi'(\rho)iz^{i-1} + \tfrac{1}{2!}\chi''(\rho)i^{\underline{2}}z^{i-2} + \ldots + \tfrac{1}{n!}\chi^{(n)}(\rho)i^{\underline{n}}z^{i-n}].$$

Therefore, as $\chi^{(j)}(\rho_k) = 0$ for $0 \le j < m_k$, it follows that $\underline{E}$ sends $\mathcal{F}$ into

$$\sum_{k=1}^r x^{\rho_k}\mathcal{O}[z]_{<m_1+\ldots+m_{k-1}} = \sum_{k=2}^r x^{\rho_k}\mathcal{O}[z]_{<m_1+\ldots+m_{k-1}} \subset \sum_{k=2}^r x^{\rho_{k-1}+1}\mathcal{O}[z]_{<m_1+\ldots+m_{k-1}} \subset x\mathcal{F}.$$

Here, we use that $\rho_k - \rho_{k-1} \in \mathbb{N}_{>0}$ and hence $\rho_{k-1} + 1 \le \rho_k$. This proves that $\underline{E}(\mathcal{F}) \subset x\mathcal{F}$.

(b) We show that $\underline{E}(\mathcal{F}) \supset x\mathcal{F}$. It suffices to check that all monomials $x^\sigma z^i \in x\mathcal{F}$ lie in the image, where $\sigma = \rho_k + e$ for some $k = 1, ..., r$ and $e \ge 1$, and where $i < m_1 + \ldots + m_k$. We distinguish two cases.

(i) If $\sigma \notin \Omega$, proceed by induction on i. Let $i = 0$. We have

$$\underline{E}(x^\sigma) = E(x^\sigma) = \chi(\sigma)x^\sigma \neq 0,$$

since σ is not a root of χ. So $x^\sigma \in \underline{E}(\mathcal{F})$. Let now $i > 0$. Lemma 3 yields

$$\underline{E}(x^\sigma z^i) = \chi(\sigma) x^\sigma z^i + \chi^{(j)}(\sigma) x^\sigma \sum_{j=1}^n \frac{i^{\underline{j}}}{j!} z^{i-j}.$$

By the inductive hypothesis and using again that $\chi(\sigma) \neq 0$, we end up with $x^\sigma z^i \in \underline{E}(\mathcal{F})$.

(ii) If $\sigma \in \Omega$, write $\sigma = \rho_k$ for some $1 \le k \le r$. As $x^\sigma z^i = x^{\rho_k} z^i \in x\mathcal{F}$ for $i < m_1 + \ldots + m_k$ and since $\rho_1 < \rho_2 < \cdots < \rho_r$, we know that $k \ge 2$ and

$$x^{\rho_k} z^i \notin x \cdot \sum_{\ell=k}^r \mathcal{O} x^{\rho_\ell}[z]_{<m_1+\ldots+m_\ell}.$$

Hence

$$x^{\rho_k} z^i \in x \cdot \sum_{\ell=1}^{k-1} \mathcal{O} x^{\rho_\ell}[z]_{<m_1+\ldots+m_\ell}.$$

This implies in particular that $0 \le i < m_1 + \ldots + m_{k-1}$, which will be used later on. We proceed by induction on i. Let $i = 0$. By Lemma 3,

$$\underline{E}(x^{\rho_k} z^{m_k}) = \sum_{j=0}^{m_k-1} \frac{(m_k)^{\underline{j}}}{j!} \chi^{(j)}(\rho_k) x^{\rho_k} z^{m_k-j} + \chi^{(m_k)}(\rho_k) x^{\rho_k} = \chi^{(m_k)}(\rho_k) x^{\rho_k}.$$

Here, the sum in the first summand is 0 since ρ_k is a root of χ of multiplicity m_k, and for the same reason, the second summand $\chi^{(m_k)}(\rho_k) x^{\rho_k}$ is non-zero. So $x^\sigma = x^{\rho_k} \in \underline{E}(\mathcal{F})$. Let now $i > 0$ and consider $x^\sigma z^i = x^{\rho_k} z^i \in x\mathcal{F}$. We will use that $i < m_1 + \ldots + m_{k-1}$ as observed above. Namely, this implies that $m_k + i < m_1 + \ldots + m_k$, so that $x^{\rho_k} z^{m_k+i}$ is an element of $\mathcal{F}$. Let us apply $\underline{E}$ to it. Similarly as in the case $i = 0$ we get

$$\underline{E}(x^{\rho_k} z^{m_k+i}) = \frac{(m_k+i)^{\underline{m_k}}}{m_k!} \chi^{(m_k)}(\rho_k) x^{\rho_k} z^i + \sum_{j=m_k+1}^n \frac{(m_k+i)^{\underline{j}}}{j!} \chi^{(j)}(\rho_k) x^{\rho_k} z^{m_k+i-j}.$$

The sum appearing in the second summand of the last line belongs to $\underline{E}(\mathcal{F})$ by the induction hypothesis since $m_k + i - j < i$. As $\chi^{(m_k)}(\rho_k) \neq 0$, we end up with $x^\sigma z^i = x^{\rho_k} z^i \in \underline{E}(\mathcal{F})$. This proves that $\underline{E}(\mathcal{F}) = x\mathcal{F}$. $\circlearrowleft$

Example. In the situation of the lemma, the image $\underline{E}(x^{\rho_1}\mathcal{O}[z]_{<m_1})$ will have a gap at $x^{\rho_2} z^i$ for $0 \le i < m_1$. It will be filled by $\underline{E}(x^{\rho_2}\mathcal{O}[z]_{<m_1+m_2})$ since $\chi^{(m_2)}(x^{\rho_2}) \neq 0$ and consequently $\underline{E}(x^{\rho_2} z^{i+m_2})$ is of the form $c_i x^{\rho_2} z^i$ plus some lower degree terms in z, for some non-zero constant $c_i \in \mathbb{C}$.

Theorem. (Normal form theorem vs3, general case) *Let $L = \sum_{j=0}^n p_j(x)\partial^j \in \mathcal{O}[\partial]$ be an n-th order linear differential operator with holomorphic coefficients p_j in $\mathcal{O}$. Let Ω be a set of local exponents of E with integer differences, ordered increasingly,*

$$\rho_1 < \rho_2 < \ldots < \rho_r,$$

meaning that $\rho_{k+1} - \rho_k \in \mathbb{N}_{>0}$. Let m_k be the multiplicity of ρ_k. Set

$$\mathcal{F} = x^{\rho_1}\mathcal{O}[z]_{<m_1} + x^{\rho_2}\mathcal{O}[z]_{<m_1+m_2} + \ldots + x^{\rho_r}\mathcal{O}[z]_{<m_1+m_2+\ldots+m_r}.$$

Denote by L_0 the initial form of L at 0, and assume that L_0 has shift 0. There exists a linear automorphism $\widehat{u} : \widehat{\mathcal{F}} \to \widehat{\mathcal{F}}$ such that the linear maps $\underline{L}$ and $\underline{L}_0$ on $\widehat{\mathcal{F}}$ induced by L and L_0 satisfy

$$\underline{L} \circ \widehat{u}^{-1} = \underline{L}_0.$$

Moreover, if 0 is a regular singular point of L, then $\widehat{u}$ restricts to a linear automorphism $u : \mathcal{F} \to \mathcal{F}$ such that the linear maps on $\mathcal{F}$ induced by $\underline{L}$ and $\underline{L}_0$ satisfy

$$\underline{L} \circ u^{-1} = \underline{L}_0.$$

Proof. Repeat the proof of version 1 of the normal form theorem, using now Lemma 5 for ensuring the required equality $\underline{L}_0(\mathcal{F}) = x\mathcal{F}$. $\circlearrowleft$

Corollary. *In the situation of the theorem and assuming that L has a regular singularity at 0, let $y_1 = x^\rho, ..., y_{m_\rho} = x^\rho \log(x)^{m_\rho - 1}$, for ρ varying over all local exponents of L, be the basis of solutions of the Euler equation $L_0 y = 0$. Let $\mathcal{F}$ be defined as above with normalizing automorphism $u : \mathcal{F} \to \mathcal{F}$. Then*

$$u^{-1}(y_1) = u^{-1}(x^\rho), ..., y_{m_\rho} = u^{-1}(x^\rho \log(x)^{m_\rho - 1})$$

form a basis of solutions of $Ly = 0$. If $\Omega = \{\rho_1, ..., \rho_r\}$ is an increasingly ordered set of local exponents with integer differences, the solutions $y_{ki} = u^{-1}(x^{\rho_k} \log(x)^i)$, for $1 \leq k \leq r$, $0 \leq i < m_k$, of $Ly = 0$ are of the form

$$y_{\rho_k i}(x) = x^{\rho_k}[f_{ki}(x) + f_{k,i-1}(x)\log(x) + \ldots + f_{k0}(x)\log(x)^i] +$$
$$+ \sum_{\ell=k+1}^{r} x^{\rho_\ell} \sum_{j=m_1+\ldots+m_{\ell-1}}^{m_1+\ldots+m_\ell - 1} h_{kij}(x)\log(x)^j,$$

with holomorphic f_{ki} and h_{kij} in $\mathcal{O}$, where all f_{ki} have non-zero constant term.

Proof. The first part is a direct consequence of the normal form theorem. The explicit description of the solutions in the second part follows by a computation from the formula $u = \mathrm{Id}_\mathcal{F} - S \circ T$ of the normalizing automorphism $u : \mathcal{F} \to \mathcal{F}$. $\circlearrowleft$

Example. We consider the operator $L = x^2\partial^2 - 2x\partial + 2 + x$ with initial form $L_0 = x^2\partial^2 - 2x\partial + 2$, shift $\tau = 0$, indicial polynomial $\chi(t) = (t-1)(t-2)$ with derivative $\chi'(t) = 2t - 3$, and local exponents $\sigma = 1$, $\rho = 2$ at 0 of multiplicity 1 each. So $\Omega = \{\sigma, \rho\} = \{1, 2\}$. Let $f(x) = \sum_{k=0}^\infty c_k x^k$, $g(x) = \sum_{k=0}^\infty a_k x^k$, $h(x) = \sum_{k=0}^\infty b_k x^k$ with $a_k, b_k, c_k \in \mathbb{C}$ be unknown holomorphic functions. The two prospective (linearly independent) solutions of $Ly = 0$ are of the form

$$y_1(x) = x^2 f(x) = \textstyle\sum_{k=0}^\infty c_k x^{k+2} \in x^2\mathbb{C}\{x\},$$
$$y_2(x) = xg(x) + x^2 h(x)\log(x)$$
$$= \textstyle\sum_{k=0}^\infty a_k x^{k+1} + \sum_{k=0}^\infty b_k x^{k+2}\log(x) \in x\mathbb{C}\{x\} \oplus x^2\mathbb{C}\{x\}\log(x).$$

The first one corresponds to the maximal exponent $\rho = 2$, the second to the exponent $\sigma = 1$. It is this second one which interests us. We set $y(x) = xg(x) + x^2h(x)z \in x\mathbb{C}\{x\} \oplus x^2\mathbb{C}\{x\}z$ and consider, according to our preceding constructions, the operator $\underline{L}$ induced by L,

$$\underline{L} : x\mathbb{C}\{x\} \oplus x^2\mathbb{C}\{x\}z \to x\mathbb{C}\{x\} \oplus x^2\mathbb{C}\{x\}z,$$
$$\underline{L}(y(x))) = (L + L'\partial_z)(y(x)) = L(xg(x)) + L'(x^2h(x)) + L(x^2h(x))z.$$

Spliting $\underline{L}$ into two components, according to the direct sum $x\mathbb{C}\{x\} \oplus x^2\mathbb{C}\{x\}z \cong x\mathbb{C}\{x\} \times x^2\mathbb{C}\{x\}$, the map $\underline{L}$ decomposes into $\underline{L} = (L^\sigma + L'^\rho, L^\rho) = (L^1 + L'^2, L^2)$ with linear maps

$$L^1 : x\mathbb{C}\{x\} \to x\mathbb{C}\{x\},$$
$$L'^2 : x^2\mathbb{C}\{x\} \to x\mathbb{C}\{x\},$$
$$L^2 : x^2\mathbb{C}\{x\} \to x^2\mathbb{C}\{x\}.$$

The analogous decompositions hold for the initial form L_0 of L. We have the formulas

$$L_0^1(x^{1+k}) = \chi(k+1)x^{1+k},$$
$$L_0'^2(x^{2+k}) = \chi'(k+2)x^{2+k},$$
$$L_0^2(x^{2+k}) = \chi(k+2)x^{2+k}.$$

The equation $Ly = 0$ is equivalent to

$$(L_0^1 + L_0'^2 + x)(xg(x), x^2h(x)) = 0$$

and

$$(L_0^2 + x)(xg(x), x^2h(x)) = 0.$$

This just means that

$$\sum_{k=0}^\infty \chi(k+1)a_kx^{k+1} + \sum_{k=0}^\infty \chi'(k+2)b_kx^{k+2} + \sum_{k=0}^\infty a_kx^{k+2} = 0,$$
$$\sum_{k=0}^\infty \chi(k+2)b_kx^{k+2} + \sum_{k=0}^\infty b_kx^{k+3} = 0,$$

say,

$$\sum_{k=0}^\infty (k^2-k)a_kx^{k+1} + \sum_{k=0}^\infty (2k+1)b_kx^{k+2} + \sum_{k=0}^\infty a_kx^{k+2} = 0,$$
$$\sum_{k=0}^\infty (k^2+k)b_kx^{k+2} + \sum_{k=0}^\infty b_kx^{k+3} = 0.$$

Reordering the sums gives

$$\sum_{k=0}^\infty ((k+1)^2-(k+1))a_{k+1}x^{k+2} + \sum_{k=0}^\infty a_kx^{k+2} + \sum_{k=0}^\infty (2k+1)b_kx^{k+2} = 0,$$
$$\sum_{k=0}^\infty ((k+1)^2+k+1)b_{k+1}x^{k+3} + \sum_{k=0}^\infty b_kx^{k+3} = 0,$$

from which we get the following system of linear recurrences ($k \geq 0$)

$$a_k + (k^2+k)a_{k+1} + (2k+1)b_k = 0,$$
$$b_k + (k^2+3k+2)b_{k+1} = 0.$$

We distinguish two cases, $b_0 = 0$ and $b_0 \neq 0$. In the first case, we get $b_k = 0$ for all $k \geq 0$, and from this follows $a_0 = 0$, $a_1 \in \mathbb{C}$ arbitrary, and

$$a_k = -\tfrac{1}{(k-1)^2+k-1}a_{k-1} = -\tfrac{1}{k^2-k}a_{k-1}$$

for $k \geq 2$. We choose $a_1 \neq 0$ in order not to get the trivial zero solution. In the second case, we may take $b_0 \in \mathbb{C}^*$ arbitrary, and then the second set of recurrences implies, for $k \geq 1$,

$$b_k = -\tfrac{1}{(k-1)^2+3(k-1)+2}b_{k-1} = -\tfrac{1}{k^2+k}b_{k-1}.$$

The first set of recurrences then implies $a_0 = -b_0 \neq 0$, $a_1 \in \mathbb{C}$ arbitrary, and, for $k \geq 2$,

$$a_k = -\tfrac{1}{(k-1)^2+k-1}[a_{k-1} + (2(k-1)+1)b_{k-1}] = -\tfrac{1}{k^2-k}[a_{k-1} + (2k-1)b_{k-1}].$$

The first case yields the solution $y_1(x) = x^2g(x)$ with g holomorphic of order 0, corresponding to the maximal exponent $\rho = 2$, the second case the solution $y_2(x) = xg(x) + x^2h(x)\log(x)$ with g and h holomorphic of order 0, corresponding to the smaller exponent $\sigma = 1$.

It may irritate here that two coefficients, namely b_0 and a_1, can be chosen arbitrarily. But in fact, varying a_1 in the expansion of $y_2(x)$ just adds a multiple of the solution $y_1(x)$ to $y_2(x)$: the recursions for a_k are the same in both cases, up to adding $\frac{2k-1}{k^2-k}b_{k-1}$ in the second case.