FUCHSIAN DIFFERENTIAL EQUATIONS: NOTES FALL 2022

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NOTES PART I: EXAMPLES

Let us present a bunch of differential equations with different types of singularities. Most of them are classical and have a geometric or number theoretic background.

The numeration follows the forthcoming Lecture Notes on Fuchsian Differential Equations. The write-up is (very) preliminary and sketchy, be cautious for errors or typos. Nothing will be proven yet. References will be added later on.

(B8 0) Differential equations with constant leading term and holomorphic coefficients have no singularities in $\mathbb{C}$. They might have singularities at ∞. [Try to find an example of such a singularity at ∞!] At all non-singular points, Cauchy's theorem applies, and not much more can be said.

(B9 1) The Euler equation $\sum_{i=0}^n c_i x^i y^{(i)} = 0$ is the prototype of an equation with a regular singularity at 0 (and at ∞). Indeed, the quotients $\frac{p_i(x)}{p_0(x)} = \frac{x^{n-i}}{x^n}$ equal x^{-i} and have all poles of exact order i at 0. All differential equations with at most two regular singularities, say, at 0 and ∞, are already Euler equations. If all exponents ρ_i are distinct, the monomials $y(x) = x^{\rho_i}$ form a C-basis of solutions. If a local exponent ρ has multiplicity m, the respective solutions are x^ρ, $x^\rho \log(x)$,..., $x^\rho \log(x)^{m-1}$.

(B10 2) The (second order) hypergeometric equation was considered already by Euler (1707-1783) and studied later extensively by Gauss (1777-1855). It has the form

$$x(x-1)y'' + ((a+b+1)x - c)y' + aby = 0,$$

with $a, b, c \in \mathbb{C}$. At first glance, the equation may seem rather arbitrary. This is not the case: on the contrary! It has three singularities, namely at 0, 1 and ∞. All three are regular. All second order linear differential equations with three regular singularities are equivalent, via an automorphism of $\mathbb{P}^1_{\mathbb{C}}$, i.e., a Möbius transformation, to the above form: take a fractional linear transformation $x \to \frac{\alpha x+\beta}{\gamma x+\delta}$, with $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{SL}_2(\mathbb{C})$ and a multiplication of the y-variable by a monomial. So the Gauss hypergeometric equation covers all these cases. The exponents at 0, 1 and ∞ are 0 and $1-c$, respectively, 0 and $c-a-b$, respectively a and b. The position of the three singularities and the values of their exponents determine the hypergeometric equation completely.

A basis of solutions of the hypergeometric equation with parameters a, b, c [excluding some special values] is given by the hypergeometric series (denote by $a^{\bar{k}} = a(a+1)\cdots(a+k-1)$ the rising factorial or Pochhammer symbol)

$$y_1(x) = {}_2F_1(a, b; c; x) = \sum_{k=0}^\infty \frac{a^{\bar{k}} b^{\bar{k}}}{c^{\bar{k}} k!} x^k,$$

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$$y_2(x) = x^{1-c}{}_2F_1(a-c+1, b-c+1; 2-c; x).$$

Idea of proof. The action of $\mathrm{SL}_2(\mathbb{C})$ on $\mathbb{P}^1_{\mathbb{C}}$ by Möbius transformation is 3-transitive: any triple of distinct points can be mapped by the action of $\mathrm{SL}_2(\mathbb{C})$ to any other triple of distinct points: If a_1, a_2, a_3 are three distinct points, apply $x \to \frac{(a_2-a_3)(x-a_1)}{(a_2-a_1)(x-a_3)}$ and obtain 0, 1, ∞ [Ince, p. 497]. Therefore, given a second order differential equation with three regular singular points, we may assume that these are located at 0, 1, and ∞. As the equation has order 2, there will be for each of these points 2 local exponents.

A prospective solution $y(x)$ of the differential equation is factored into $y(x) = x^{-s_0}(x-1)^{-s_1} \cdot z(x)$. This yields for $z(x)$ a new differential equation with the same singularities but modified exponents $(0, \rho_0 - \sigma_0)$, $(0, \rho_1 - \sigma_1)$ and $(\rho_\infty + \sigma_0 + \sigma_1, \sigma_\infty + \sigma_0 + \sigma_1)$. Thus the scheme of exponents has become the same as the one for the hypergeometric equation. As the location of the singularities and their local exponents determine the differential equation, we are done.

In [Ince, p. 496] a similar procedure is applied (for second order equations): Assume that $a \in \mathbb{C}$ is a regular singularity with exponents ρ and $\rho + \frac{1}{2}$. Multiply the variable y in the differential equation by $(x-a)^{-\rho}$, set $z = (x-a)^{-\rho}y$, and get a differential equation for z whose exponents are now 0 and $\frac{1}{2}$ [please check this!]. This can be done simultaneously for all singularities in $\mathbb{C}$, regardless of the type of the singularity at ∞, whose exponents do not change under the transformation. Of course, from a solution $z(x)$ of the new differential equation the solution of the original equation can easily be reconstructed via $y(x) = (x-a)^\rho z(x)$.

Landau indicates instances already observed by Schwarz for the algebraicity of all the solutions expressed in terms of the parameters a, b, c and the differences $c-a$, $c-b$. He uses Eisenstein's theorem to deduce these conditions in a mostly computational manner, see also [Höpp].

(B10' 2') The general hypergeometric equation. Write an n-th order differential operator L as an operator in $\delta = x\partial_x$,

$$L_\delta = \delta^n + q_1(x)\delta^{n-1} + \ldots + q_{n-1}(x)\delta + q_n(x),$$

with $q_i \in \mathbb{C}(x)$ rational functions. It has regular singularities in 0, 1 and ∞ and is non-singular elsewhere if and only if [Beukers-Heckman, Prop. 2.1, p. 327]

$$q_i(x) = \textstyle\sum_{j=0}^{i} c_{ij}(x-1)^{-j}, \qquad \text{for } c_{ij} \in \mathbb{C}.$$

It is called *hypergeometric* if

$$q_i(x) = c_{i0} + c_{i1}(x-1)^{-1}$$

for all i, i.e., if the poles of q_i at $x = 1$ have at most order 1. In this case, one may factor $(1-x)L$ into [Beukers-Heckman, p. 327]

$$(1-x)L = (\delta + \beta_1 - 1)\cdots(\delta + \beta_n - 1) - x(\delta + \alpha_1)\cdots(\delta + \alpha_n)$$

with $\alpha_i, \beta_i \in \mathbb{C}$. The local exponents at 0, 1 and ∞ are $1-\beta_1, ..., 1-\beta_n$ at $x = 0$, $\alpha_1, ..., \alpha_n$ at $x = \infty$, and $0, 1, ..., n-2$ and $\sum_{i=1}^n \beta_i - \sum_{i=1}^n \alpha_i$ at $x = 1$. If $\beta_1, ..., \beta_n$ are pairwise not congruent modulo $\mathbb{Z}$, a basis of solutions of $Ly = 0$ is given by

$$y_i(x) = x^{1-\beta_i}{}_nF_{n-1}(1+\alpha_1-\beta_i, ..., 1+\alpha_n-\beta_i; 1+\beta_1-\beta_i, ..., \widehat{1}, ..., 1+\beta_n-\beta_i; x),$$

where $1 + \beta_i - \beta_i = 1$ is omitted and where

$$ {}_nF_{n-1}(a_1, ..., a_n; b_1, ..., b_{n-1}; x) = \sum_{k=0}^{\infty} \frac{a_1^{\bar{k}} \cdots a_n^{\bar{k}}}{b_1^{\bar{k}} \cdots b_{n-1}^{\bar{k}} n!} x^k. $$

(B11 3) Here is an example of a (regular) singular differential equation whose solutions are nevertheless nice: Take $x^2y'' - 3xy' + 3y = 0$. This is an Euler equation. The indicial polynomial is $\rho(\rho-1) - 3\rho + 3 = \rho^2 - 4\rho + 3 = (\rho - 1)(\rho - 3)$. The solutions are spanned by $y_1(x) = x$, $y_2(x) = x^3$, hence holomorphic despite the presence of the singularity. The singularities of differential equations which admit a basis of locally holomorphic solutions are called *apparent singularities*. They are kind of "harmless".

(B12 4) In general, the local solutions of a regular singular differential equation are no longer power series (not even formal ones). Take $x^2y'' - xy' + y = 0$ with indicial polynomial $(\rho - 1)^2$ and solutions x and $x\log(x)$. And the equation $xy' - \alpha y = 0$ has local solution cx^α, $c \in \mathbb{C}$, for any $\alpha \in \mathbb{C}$. For $\alpha \notin \mathbb{Z}$, this defines a "multivalued" function $x^\alpha = \exp(\alpha \log(x))$ at 0.

(B13 5) The Legendre differential equation

$$ 4t(t-1)z'' + 4(2t-1)z' + z = 0 $$

is associated to the family of elliptic curves $E_t : y^2 = x(x-1)(x-t)$, $t \in \mathbb{C}$, by integrating the (essentially) unique holomorphic 1-form

$$ \omega_t = \frac{dx}{y} = \frac{dy}{(x(x-1)(x-t))^{1/2}} $$

on E_t. Then the integral $z(t) = \int_\gamma \omega_t$ of ω_t along a closed path γ on E_t satisfies the differential equation [...the path has to be varied continuously with t, but this does not affect the integral]. The equation has clearly regular singularities at 0 and 1, but what about ∞?

(B14 6) The Bessel equation is $x^2y'' + xy' + (x^2 - \alpha^2)y = 0$ with $\alpha \in \mathbb{C}$. For $\alpha \neq 0$, it has a regular singularity at 0. At ∞, the transformed equation is $x^4y'' + x^3y' + (1 - \alpha^2x^2)y = 0$, hence ∞ is an irregular singularity of the Bessel equation.

Exponents at 0 are $\pm\alpha$, first local solution $y_1(x) = x^\alpha \sum_{i=0}^\infty c_i x^i$, $c_0 = 1$, with linear recursion $i(i + 2\alpha)c_i + c_{i-2} = 0$, $c_i = 0$ for i odd. This is the Bessel function. Second solution (for $\alpha \neq 0$) is more complicated and involves harmonic numbers $h_j = \sum_{k=1}^i \frac{1}{k}$ and the Euler-Mascheroni constant $\gamma = \lim_{i\to\infty}(h_i - \log(i)) = 0.577216...$; it is of the form $y_2(x) = x^{-n}z(x) + c\log(x)y_1(x)$. The case $\alpha = 0$ has to be treated separately.

The Bessel functions arise naturally when solving the Poisson equation for a system with cylindrical symmetry.

[physics stackexchange: https://physics.stackexchange.com/questions/145177/what-physical-phenomena-are-modelled-by-chebyshev-equation]

(B15 7) Apéry's differential equation is of the form

$$ (x^4 - 34x^3 + x^2)y''' + (6x^3 - 153x^2 + 3x)y'' + (7x^2 - 112x + 1)y' + (x - 5)y = 0. $$

It has four regular singularities, at 0, ∞ and $(1 \pm \sqrt{2})^4$ (see example (P7)). The associated linear recursion has order 2, with cubic coefficients,

$$ k^3c_k = (34k^3 - 51k^2 + 27k - 5)c_{k-1} - (k-1)^3c_{k-2}. $$

For initial values $c_0 = 1$ and $c_1 = 5$ one obtains integer values $c_k = \sum_{i=0}^k \binom{k+i}{i}^2 \binom{k}{i}^2$. For initial values $c_0 = 0$ and $c_1 = 6$ one obtains only that $\mathrm{lcm}(1, 2, ..., k)^3 c_k$ is integral, while c_k itself is not globally bounded (lcm denotes the lowest common multiple).

As a matter of curiosity, the square-root $\sqrt{y(x)}$ of a solution to Apéry's equation satisfies a differential equation of second order, namely

$$(x^3 - 34x^2 + x)y'' + (2x^2 - 51x + 1)y' + \tfrac{1}{4}(x - 10)y = 0.$$

One says that Apéry's equation is the *square* of the latter equation. The second order equation has the same four regular singularities, at 0, ∞, and $(1 \pm \sqrt{2})^4$. The respective linear recursion is

$$k^2 c_k = (34k^2 - 51k + 39/2)c_{k-1} - (k - 3/2)^2 c_{k-2}.$$

(B16 8) The Airy equation $y'' - xy = 0$ (George Biddell Airy, 1801-1892, article on optics from 1838) has a unique singular point, namely at ∞. The local form at ∞ corresponds to the equation $x^5 y'' + 2x^4 y' - y = 0$ at 0. Setting $Y = (y, y')^T$, we get the equivalent system of first order linear differential equations

$$Y' = \begin{pmatrix} 0 & 1 \\ x^{-5} & -2x^{-1} \end{pmatrix} \cdot Y$$

representing the Airy equation at ∞. A fundamental matrix of solutions of this system (now considered at 0) is

$$Y(x) = \Phi(x) x^J U e^{Q(\sqrt{x})}$$

with

$$U = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, J = \begin{pmatrix} 1/4 & 0 \\ 0 & -3/4 \end{pmatrix}, Q = \begin{pmatrix} -2/3x^{3/2} & 0 \\ 0 & 2/3x^{3/2} \end{pmatrix},$$

and some function $\Phi(x)$. Outside ∞, the local solutions are surprisingly complicated: [Mahaffy] gives as solutions at 0 the expansions

$$\begin{aligned} y(x) &= c_0 \sum_{k=0}^{\infty} \frac{1}{2 \cdot 3 \cdot 5 \cdot 6 \cdots (3k-1)3k} x^{3k} + c_1 \sum_{k=0}^{\infty} \frac{1}{3 \cdot 4 \cdot 6 \cdot 7 \cdots 3k(3k+1)} x^{3k+1} \\ &= c_0 Ai(x) + c_1 Bi(x) \end{aligned}$$

with the Airy functions

$$Ai(x) = \frac{1}{\pi} \int_0^{\infty} \cos(\frac{1}{3} z^3 + xz) dz,$$

$$Bi(x) = \frac{1}{\pi} \int_0^{\infty} \exp(-\frac{1}{3} z^3 + xz) + \sin(\frac{1}{3} z^3 + xz) dz,$$

For a graphical presentation, see [Mahaffy, p. 8]. Both functions oscillate on the negative real axis, while on the positive real axis Ai tends to 0 and Bi to ∞. Airy's equation is related to the one-dimensional time independent Schrödinger equation with total energy E

$$-\frac{\hbar}{2m} y'' + V(x) y = Ey.$$

This equation becomes for the special potential $V(x) = x$ the (modified) Airy equation

$$y'' - \frac{2m}{\hbar}(x - E) y = 0.$$

(B17 9) The operators $L_1 = x^2\partial^2 - x\partial - x^3$, $L_2 = x^2\partial^2 - x\partial - x^2$ and $L_3 = x^2\partial^2 - x\partial - x$ have the same initial form $\mathrm{in}(L_i) = x^2\partial^2 - x\partial$ but show quite different behaviour when one tries to find normal forms for them and to compute their power series solutions [Gann-Hauser, ex. 1, 1^{bis}, 1^{ter}, p. 14].

(B18 10) Here are four second order differential equations with four regular singular points admitting at least one power series solution with integral coefficients [ChCh2, p. 20],

$$x(x^2-1)y''+(3x^2-1)y'+xy=0.$$

$$x(x^2+3x+3)y''+(3x^2+6x+3)y'+(x+1)y=0.$$

$$x(x-1)(x+8)y''+(3x^2-14x-8)y'+(x+2)y=0.$$

$$x(x^2+11x-1)y''+(3x^2+22x-1)y'+(x+3)y=0.$$

In general, it seems to be extremely difficult to detect from the differential equation whether there exists a solution with integer coefficients (for a suitable choice of initial values). Apéry's equation is such an example. Zagier made a whole search for further examples. Among a 100 million computed cases of Apéry type equations, he found only seventeen equations with integral solutions.

(B19 11) $(x^2-b^2)(x^2-c^2)y''+x(x^2-b^2+x^2-c^2)y'-[m(m+1)x^2-(b^2+c^2)p]y=0$ Lamé's equation.

(B20 12) Legendre's equation, with eigenvalue λ. Solutions can be extended into singularity if and only if $\lambda=n(n+1)$, and the solutions are then the associated Legendre polynomials. The equation arises naturally when solving the Poisson equation for a system with spherical symmetry (such as the hydrogen atom). Legendre's equation occurs quite often in areas such as electrodynamics and quantum mechanics.

[physics stackexchange: https://physics.stackexchange.com/questions/145177/what-physical-phenomena-are-modelled-by-chebyshev-equation]

(B21 13) $(x^2-1)y''+xy'-\lambda^2y=0$ Chebyshev's equation. Regular singularities at ±1 and ∞. Recursion $c_{i+2}=\frac{i^2-\lambda^2}{(i+2)(i+1)}c_i$. Solutions involve Chebyshev polynomials of first and second kind.

[physics stackexchange: https://physics.stackexchange.com/questions/145177/what-physical-phenomena-are-modelled-by-chebyshev-equation]

(B22 14) $y''-2xy'+\lambda y=0$,

Hermite's equation. Irregular singularity at ∞. Solutions at 0 are holomorphic, can be expressed as linear combinations of two hypergeometric series, the second being the Hermite polynomial H_n if $\lambda=2n\in2\mathbb{N}$. The recursion for the coefficients is $c_{i+2}=\frac{2i-\lambda}{(i+2)(i+1)}c_i$.

[https://mathworld.wolfram.com/HermiteDifferentialEquation.html]

$H_1(x)=2x$, $H_2(x)=4x^2-2$, $H_3(x)=8x^3-12x$, ...

Exponential generating function: $\sum_{k=1}^\infty H_k(x)\frac{1}{k!}t^k=e^{2xt-t^2}$.

(B23 15) $xy''+(\nu+1-x)y'+\lambda y=0$,

Laguerre's equation. Regular singularity at 0 and irregular singularity at ∞. If $\lambda\in\mathbb{N}$, the solution at 0 is polynomial and thus extends into 0, giving the associated Laguerre polynomial for arbitary n, and the (classical) Laguerre polynomial for $\nu=0$. The recursion for the coefficients is $c_{i+1}=\frac{i-\lambda}{(i+1)(i+\nu+1)}c_i$.

[Wolfram: https://archive.lib.msu.edu/crcmath/math/math/l/l039.htm]

Physics Forum: The equation arises in solving Schrödinger's equation to find the quantum-mechanical wave function of hydrogen. Specifically, it's associated with the radial part of the wave function.

[https://www.physicsforums.com/threads/uses-of-laguerre-differential-equ.222949/]

(B24 16) $xy'' + (c - x)y' - ay = 0$,

Kummer's equation (confluent hypergeometric equation). Has regular singularity at 0 and irregular singularity at ∞. It is called *confluent* since in the hypergeometric equation with three singular points two are merged to one singularity.

[https://math.stackexchange.com/questions/190486/transforming-differential-equation-to-a-kummers-equation]

(B25 17) $(1 - x^2)y'' - 2(\mu + 1)xy' + (\nu - \mu)(\nu + \mu + 1)y = 0$,

Gegenbauer's equation, singularities at ± 1. If $-1/2 + \mu + \nu$ is an integer n, one of the solutions is the Gegenbauer polynomial $C_n(x)$.

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NOTES PART II: BASICS

Singularities of differential equations. A point $a \in \mathbb{P}^1_{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ is called a *singularity* of a monic complex linear differential equation with meromorphic coefficients $Ly = y^{(n)} + p_1 y^{(n-1)} + \cdots + p_n y = 0$ if at least one of the coefficients $p_i(x)$ has a pole at a. If all p_i are holomorphic at a, then a is a *non-singular* point of L. A singularity is called *regular* if it has a basis of local solutions which have moderate growth as one approaches the singular point. This is for the moment just an intuitive definition which still has to be made precise (also, the concept of basis of solutions is not specified rigorously yet). We will show later that regularity is equivalent to saying that for every $i = 1, ..., n$, the order of the pole of $p_i(x)$ at a is at most i. A singular point which is not regular is called an *irregular singularity.*

A point $a \in \mathbb{C} \cup \{\infty\}$ is called *apparent singularity* of L if $Ly = 0$ possesses locally at a a basis of holomorphic solutions. In example 4 from above, 0 is an apparent singularity, since the local solutions at 0 are polynomials.

Let $L = \sum_{j=0}^n p_j \partial^j = \sum_{j=0}^n \sum_{i=0}^\infty c_{ij} x^i \partial^j$ be a differential operator with polynomial, holomorphic or formal coefficients $p_j(x) = \sum_{i=0}^\infty c_{ij} x^i$. The *initial form* (or initial operator) of L at 0 is the operator

$$L_0 = \mathrm{in}(L) = \textstyle\sum_{i-j=\tau} c_{ij} x^i \partial^j,$$

where $\tau = \min_{c_{ij} \neq 0}\{i - j\} \in \mathbb{Z}$ is the *shift* of L (or rather of L_0) at 0. The polynomial

$$\chi_L(r) = \textstyle\sum_{i-j=\tau} c_{ij} r^{\underline{j}},$$

where $r^{\underline{j}} = r(r-1)\cdots(r-j+1)$ denotes the *falling factorial*, is called the *indicial* or *characteristic polynomial* of L at 0. Cleary, $\chi_L = \chi_{L_0}$. Analogous definitions hold at other points $a \in \mathbb{P}^1_{\mathbb{C}}$, taking into account the respective Taylor expansions of the coefficients of L at a, replacing x by $x - a$ if $a \in \mathbb{C}$ and leaving ∂ invariant. For $a = \infty$, one has to replace in the coefficients of L the variable x by $\frac{1}{x}$ and adjust accordingly the derivations ∂^i, using the differentiation rules for $\partial^i[f(\frac{1}{x})]$. In particular, ∂ has to be replaced by $-\frac{1}{x^2}\partial$. The resulting operator has then to be considered at 0 (see below).

If a is a non-singular point or a regular singularity of the differential operator L, then the complex roots of the indicial polynomial χ_L of L at a are called the *local exponents* of L at a.

Fact. *The point 0 is a non-singular point or a regular singularity of L if and only if the initial form L_0 of L has the same order as L.*

Proof. The assertion is easily checked using the order condition on the poles from above. An extensive characterization of regular singularities will be provided later on. $\circlearrowleft$

Expansion at infinity. If $L = \sum_{j=0}^n \sum_{i=0}^\infty c_{ij} x^i \partial^j$ is a differential operator on $\mathbb{P}^1_{\mathbb{C}}$ with meromorphic coefficients p_i we may expand L in a local chart at ∞. To this end, replace x by $\frac{1}{x}$ in the coefficients $p_i(x)$ and in the solutions $y(x)$ of $Ly = 0$. Let us call ψ this automorphism of $\mathbb{P}^1_{\mathbb{C}}$. Applying ψ results in a change of the differential operator L to an operator $\psi^*(L)$ – the *pullback* of L under ψ – where now also the derivations ∂^i will have to be adapted. In fact, for a function f, the i-th derivative $\partial^i[f(\frac{1}{x})] = \partial^i[f(\psi(x))]$ can be expressed as the composition of a differential operator $L_i = \psi^*(\partial^i)$ applied to $f(x)$ with the subsequent substitution of x by $\frac{1}{x}$, say, $\partial^i[f(\frac{1}{x})] = (L_i f)(\frac{1}{x})$. In particular, we will have

$$L_1 = -\frac{1}{x^2}\partial, \qquad \text{since } \partial[f(\tfrac{1}{x})] = -\tfrac{1}{x^2}(\partial f)(\tfrac{1}{x}),$$

$$L_2 = \frac{1}{x^4}\partial^2 + 2\frac{1}{x^3}\partial, \qquad \text{since } \partial^2[f(\tfrac{1}{x})] = \tfrac{1}{x^4}(\partial^2 f)(\tfrac{1}{x}) + 2\tfrac{1}{x^3}(\partial f)(\tfrac{1}{x}),$$

$$L_3 = -\frac{1}{x^6}\partial^2 - 6\frac{1}{x^5}\partial^2 - 6\frac{1}{x^4}\partial,$$

$$\text{since } \partial^3[f(\tfrac{1}{x})] = -\tfrac{1}{x^6}(\partial^2 f)(\tfrac{1}{x}) - 4\tfrac{1}{x^5}(\partial^2 f)(\tfrac{1}{x}) - 2\tfrac{1}{x^5}(\partial^2 f)(\tfrac{1}{x}) - 6\tfrac{1}{x^4}(\partial f)(\tfrac{1}{x}).$$

Example. (I11) Let us take the operator $L = x^3\partial^2 - (x - x^2)\partial + 1$. It has initial form $L_0 = -x\partial + 1$ at the origin 0 of $\mathbb{C}$. As its order is smaller than the order of L, the point 0 is an irregular singularity of L. For instance, $y(x) = \sum_{k=0}^{\infty} k!x^{k+1}$ is a divergent formal power series solution of $Ly = 0$. Let us compute the expansion of L at ∞. Substitution gives

$$\psi^*(L) = \frac{1}{x^3}\frac{1}{x^4}\partial^2 + 2\frac{1}{x^3}\frac{1}{x^3}\partial - (\frac{1}{x} - \frac{1}{x^2})(-\frac{1}{x^2})\partial + 1 =$$
$$\frac{1}{x^7}\partial^2 + [2\frac{1}{x^6}\partial + \frac{1}{x^3} - \frac{1}{x^4}]\partial + 1.$$

Multiplication with the common denominator x^7 results in the operator

$$\widetilde{L} = \partial^2 + [2x + x^4 - x^3]\partial + x^7,$$

which is non-singular at 0. This shows that the local structure of a differential equation at ∞ may not be immediately obvious from the expansion of the operator at 0.

Systems of linear differential equations. Let (K, ∂) be a differential field, i.e., a field together with a derivation $\partial : K \to K$ (think of K the field of meromorphic functions and $\partial = \frac{d}{dx}$ the usual derivative). The field of constants $C \subset K$ consists of the elements f with $\partial f = 0$. A system of n linear first order equations over K is of the form

$$\begin{array}{lll} y_1' & = a_{11}y_1 + \cdots & + a_{1n}y_n \\ \vdots & \quad\vdots & \quad\vdots \\ y_n' & = a_{n1}y_1 + \cdots & + a_{nn}y_n \end{array}$$

or, in matrix notation, $\partial Y = AY$, with the unknown column vector $Y = (Y_1, Y_2, ..., Y_n)^T$ and an $(n \times n)$ matrix $A = (a_{ij}) \in \mathrm{M}_n(K)$. Here ∂ acts on K^n componentswise, i.e., $\partial(Y_1, ..., Y_n)^T = (\partial Y_1, ..., \partial Y_n)^T$. The induced linear map is

$$L = \partial - A : K^n \to K^n,$$
$$Y \to \partial Y - AY.$$

If $K \subset K'$ is a field extension with $\partial' : K' \to K'$ extending $\partial : K \to K$, any vector $Y \in K'^n$ such that $\partial Y = AY$ is called a solution of $\partial Y = AY$ in K'. An $(n \times n)$ invertible matrix $\Phi \in \mathrm{GL}_n(K')$ with $\Phi' = A\Phi$ is called a *fundamental solution matrix* of $Y' = AY$ in K'.

Remark. We will see later that the set $\mathrm{Sol}_K := \{Y \in K^n, \partial Y = AY\}$ is a vector space of dimension $\leq n$ over the field of constants C of K. In general, $\dim_C(\mathrm{Sol}_K) < n$. However, there always exists a differential field extension $K \subset K'$ such that over K' the solution space has dimension n. Such extensions are known, if they are minimal, as Picard-Vessiot extensions [vdPS].

Example. (I13) Let K be one of the fields $\mathbb{C}\{\{x\}\} = \mathrm{Quot}(\mathbb{C}\{x\})$ or $\mathbb{C}((x)) = \mathrm{Quot}(\mathbb{C}[[x]])$ equipped with the derivation $\partial : K \to K$ defined by $\partial x = 1$ and set $\delta = x\partial : K \to K$, $\delta x = x$. If $A \in \mathrm{M}_n(K)$, we obtain the maps

$$\partial - A : K^n \to K^n,$$

$$Y \mapsto Y' - AY,$$

and

$$\delta - A : K^n \to K^n,$$

$$Y \mapsto xY' - AY.$$

For example, consider the system $Y' = AY$ with $A = \begin{pmatrix} 0 & 1 \\ \frac{-2}{x} & \frac{x+2}{x} \end{pmatrix}$. We have $Y' = AY$ if and only if $y_1' = y_2$ and $y_2' = -\frac{2}{x}y_1 + \frac{(x+2)}{x}y_2$. This gives, setting $y = y_1$ and $y' = y_2$ [exercise: check the formulas] the scalar equation $y'' = -\frac{2}{x}y + \frac{(x+2)}{x}y'$, say $xy'' - (x+2)y' + 2y = 0$. Then $y_1(x) = e^x$ and $y_2(x) = 1 + x + \frac{1}{2}x^2$ are $\mathbb{C}$-linearly independent solutions of this equation. Therefore

$$\begin{pmatrix} e^x \\ e^x \end{pmatrix}$$

and

$$\begin{pmatrix} 1 + x + \frac{1}{2}x^2 \\ 1 + x \end{pmatrix}$$

are linearly independent solutions of $Y' = AY$ and hence $\Phi = \begin{pmatrix} e^x & 1 + x + \frac{1}{2}x^2 \\ e^x & 1 + x \end{pmatrix}$ is a fundamental solution matrix of the system $Y' = AY$.

The *singularities* of the system $\partial Y = AY$ are the poles of the entries of A. Similarly as for scalar equations, a singularity of a system is called *apparent* if there exists a fundamental solution matrix $\widetilde{Y}(x)$ of $\partial Y = AY$ with holomorphic entries.

Remark. Note that if we replace Y by PY in the system, where $P \in \mathrm{GL}_n(K)$ is an invertible matrix, we obtain a new system

$$\partial Y = (P^{-1}AP - P^{-1}\partial P)Y =: BY,$$

with $B = P^{-1}AP - P^{-1}\partial P$. Two systems $\partial Y = AY$ and $\partial Y = BY$ are called *gauge equivalent* (over K) if there exists $P \in \mathrm{GL}_n(K)$ so that $B = P^{-1}AP - P^{-1}\partial P$. If $P \in \mathrm{GL}_n(K')$ for some differential field extension K' of K then $\partial Y = AY$ and $\partial Y = BY$ are called gauge equivalent over K'.

Expressed in terms of maps we get from $(P^{-1} \circ \partial \circ P)(Y) = P^{-1}(\partial(PY)) = P^{-1}(\partial P)Y + P^{-1}P\partial Y = P^{-1}(\partial P)Y + \partial Y = (P^{-1}\partial P + \partial)Y$ that

$$\begin{aligned}
\partial - B = P^{-1} \circ (\partial - A) \circ P &= P^{-1} \circ \partial \circ P - P^{-1} \circ A \circ P \\
&= P^{-1}\partial P + \partial - P^{-1}AP \\
&= \partial - (P^{-1}AP - P^{-1}\partial P) \\
&= P^{-1}AP - P^{-1}\partial P.
\end{aligned}$$

Lemma (Jósef Maria Hoëné-Wroński, 1776-1853) *Let be given n holomorphic functions $y_1, ..., y_n$ defined in a neighborhood of $0 \in \mathbb{C}$. They are $\mathbb{C}$-linearly dependent if and only if the* Wronskian matrix

$$W(y_1, ..., y_n) = \begin{pmatrix} y_1 & \cdots & y_n \\ y_1' & \cdots & y_n' \\ \vdots & & \vdots \\ y_1^{(n-1)} & \cdots & y_n^{(n-1)} \end{pmatrix}$$

formed by the row vector $(y_1, ..., y_n)$ and its first $n-1$ derivatives has zero determinant.

Remarks. (a) This can be done as an exercise, using induction on n, see below. See also [Honda, Lemma, p. 172, resp. Kol1] for a characteristic p version: If the determinant of Wronskian vanishes, then $y_1, ..., y_n$ are linearly dependent over $K(x^p)$. See also appendix B in [diVi2].

(b) The case $n = 2$ is particularly instructive. Let $y = y(x)$ and $z = z(x)$ be two holomorphic functions, not identically 0, and assume that their Wronskian determinant is 0,

$$\det \begin{pmatrix} y & z \\ y' & z' \end{pmatrix} = yz' - y'z = 0.$$

This is equivalent to $\frac{y}{y'} = \frac{z}{z'}$ and thus to $\log(y)' = \log(z)'$. We get equivalently $\log y = \log z + c$ for some constant $c \in \mathbb{C}$, hence $y = e^c \cdot z$ as claimed.

Proof. We prove the non-trivial implication. It is a bit tricky. So let $W(y_1, ..., y_n)$ have zero determinant. If $n = 1$, we get $y_1 = 0$. We now assume $n \geq 2$, and wlog that $y_n \neq 0$. One checks by computation that

$$\det(W(y_1, ..., y_n)) = y_n^n \cdot \det(W(y_1/y_n, ..., y_{n-1}/y_n, 1)),$$

and

$$\det(W(y_1/y_n, ..., y_{n-1}/y_n, 1)) = (-1)^n \cdot \det(W((y_1/y_n)', ..., (y_{n-1}/y_n)')).$$

Applying the lemma in the case $n-1$ we get constants $c_1, ..., c_{n-1} \in \mathbb{C}$ such that

$$c_1 \cdot (y_1/y_n)' + \cdots + c_{n-1} \cdot (y_{n-1}/y_n)' = 0.$$

It follows that

$$c_1 \cdot y_1/y_n + \cdots + c_{n-1} \cdot y_{n-1}/y_n = c,$$

for some $c \in \mathbb{C}$. This shows that $y_1, ..., y_n$ are $\mathbb{C}$-linearly dependent. ⥁

Corollary. *An n-th order linear differential equation $Ly = 0$ with holomorphic coefficients has at most n $\mathbb{C}$-linearly independent local holomorphic solutions.*

Proof. Assume we had $n+1$ solutions $y_1, ..., y_{n+1}$. The columns of $W(y_1, ..., y_{n+1})$ are given by $(y_i, y_i', ..., y_i^{(n-1)}, y_i^{(n)})^T$. The entries of each of these vectors are $\mathbb{C}\{x\}$-linearly dependent since they satisfy the linear relation given by $Ly = 0$. It follows that the determinant of $W(y_1, ..., y_{n+1})$ is zero. By Wronski's lemma we conclude that $y_1, ..., y_{n+1}$ are $\mathbb{C}$-linearly dependent. ⥁

Lemma. *Consider two n-th order linear differential equations $Ly = 0$ and $My = 0$. Assume given holomorphic functions $y_1, ..., y_n$ at 0 which form a basis of solutions for both L and M. Then there exists a meromorphic function h at 0 such that $M = h \cdot L$.*

Proof. It is sufficient to prove the assertion for formal power series operators. The convergent case goes along the same lines. Let $\mathbb{C}((x))$ denote the quotient field of $\mathbb{C}[[x]]$, i.e., the field of formal Laurent series. Define a map $\alpha : \mathbb{C}((x))[\partial] \to \mathbb{C}((x))^n$, sending a differential operator N to the vector $(Ny_1, ..., Ny_n)$ given by evaluation. By definition, L and M belong to the kernel of α. But $\mathbb{C}((x))[\partial]$ is a polynomial ring over a field and hence a principal ideal domain. Hence $\mathrm{Ker}(\alpha)$ is generated by one operator N, and L and M are $\mathbb{C}((x))[\partial]$-multiples of it. But $y_1, ..., y_n$ are then also $\mathbb{C}$-linearly independent solutions of N, therefore N has order at least n. As L and M are multiples of it (as elements of the ring $\mathbb{C}((x))[\partial]$), N must have order n. This implies that $L = f \cdot N$, $M = g \cdot N$ for suitable $f, g \in \mathbb{C}((x))$. Setting $h = f/g$ we get $M = g \cdot L$ as required. ⥁

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NOTES PART III: EULER EQUATIONS

The objective of this section is to describe the local solutions of Euler differential equations $L_0 y = 0$ at 0 in the case where a local exponent ρ is a $multiple$ root of the indicial polynomial χ_{L_0}.

The situation for general linear differential equations $Ly = 0$ (always assuming the singularities to be regular) will be discussed in a later chapter. The construction of solutions of $Ly = 0$ goes back to Fuchs and Frobenius. The latter cites in [Frob1] two papers of Fuchs as predecessors of his investigation [Fuchs1, Fuchs2] as well as a paper of Thomé [Thom1], p. 200, shortening the proof of Fuchs, see also [Thom2, Thom3]. In [Mez, p. 58] the author attributes the first description of the solutions and the use of variation of constants to Fuchs, while Frobenius improved and simplified Fuchs' construction by treating the solutions involving logarithms directly. See [Gray, J.: Linear differential equations and group theory from Riemann to Poincaré. Birkhäuser, 2000] for a historical account.

Ince [Ince, footnote, p. 396] reproduces quite accurately their methods, see also section 4.3 in [Mez]. You may also consult [Teschl, section 4.4, p. 134] for an exposition of Frobenius' method. Mezzarobba presents also another method to construct solutions, developed apparently by Heffter in 1894 and exposed in the book of Poole from 1936, see [Mez, section 4.4, Poole, V.16]. We will present here a slightly modernized version of the Frobenius story. Some astonishing turns will enrich our journey.

It turns out that from now on logarithms will appear. As these are no longer holomorphic at 0 and thus do not admit a power series expansion, we have to enlarge our space of functions $\mathbb{C}\{x\}$, $\mathbb{C}[[x]]$ or $\mathbb{C}((x))$ so as to include also powers $\log(x)^k$. We do this universally by adjoining a variable z which will mimic the role of $\log(x)$. In the sequel, $\mathbb{C}((x))$ will denote a field of formal Laurent series with monomials whose exponents may even be complex numbers. In order to have a well defined multiplication, we restrict to the field generated by series of the form $h = x^\rho \sum_{i=0}^\infty c_i x^i$ with $\rho \in \mathbb{C}$ and $i \in \mathbb{N}$. We will neglect in this section convergence questions and only work formally. Monomials x^ρ with complex exponents $\rho \in \mathbb{C}$ will not do any harm: differentiation is defined as usual, $\partial x^\rho = \rho x^{\rho-1}$. Integration is given by $\int x^\rho = \frac{1}{\rho+1} x^{\rho+1}$ provided that $\rho \neq -1$.

We denote by $\mathbb{C}((x))[z]$ the ring of polynomials in a new variable z with coefficients in $\mathbb{C}((x))$. Working with $\mathbb{C}((x))[z]$ instead of the polynomial ring $\mathbb{C}((x))[\log(x)]$ in $\log(x)$ has notational advantages – the substance is of course the same. Compare with page 184 in [Honda, T.: Algebraic differential equations, Symp. Math. 24, 169-204. Academic Press 1981]. We consider $\mathbb{C}((x))[z]$ as a differential ring via the derivation

$$\underline{\partial} : \mathbb{C}((x))[z] \to \mathbb{C}((x))[z],$$

$$\underline{\partial}(x^i) = \partial(x^i) = i x^{i-1}, \qquad \underline{\partial} z = x^{-1},$$

$$\underline{\partial}(x^i z^k) = (iz + k) x^{i-1} z^{k-1}.$$

This construction formalizes the differential ring $\mathbb{C}((x))[\log(x)]$ equipped with the usual differentiation operator ∂. We shall call $\underline{\partial}$ the $logarithmic\ extension$ of ∂ to $\mathbb{C}((x))[z]$. We claim to have an isomorphism of differential rings

$$((\mathbb{C}((x))[z], \underline{\partial}) \to \mathbb{C}((x))[\log(x)], \partial),\ z \to \log(x).$$

Indeed, the map is linear, surjective and, as $\partial(\log(x)) = (\log(x))' = x^{-1}$, compatible with the derivations $\underline{\partial}$ and ∂. It is also injective since a relation $\sum_{k=0}^m h_k(x)\log(x)^k = 0$ with $h_k \in \mathbb{C}((x))$ implies that all h_k are 0. This last part is due to the fact known from analysis that $\log(x)$ is transcendental over $\mathbb{C}((x))$, i.e., does not satisfy any polynomial relation with coefficients in $\mathbb{C}((x))$. We have the differentiation rule

$$\underline{\partial}(\textstyle\sum_{k=0}^m h_k(x)z^k) = \sum_{k=0}^m \partial h_k(x)z^k + \sum_{k=1}^m kh_k(x)x^{-1}z^{k-1}.$$

Note that $\underline{\partial}$ does not increase the degree in z of polynomials in $\mathbb{C}((x))[z]$. If we denote by $\mathbb{C}((x))[z] = \bigoplus_{k=0}^\infty \mathbb{C}((x))[z]_k$ the natural grading defined by the degree in z, we get by restriction maps

$$\underline{\partial} : \mathbb{C}((x))[z]_k \to \mathbb{C}((x))[z]_k \oplus \mathbb{C}((x))[z]_{k-1}.$$

We may thus write

$$\underline{\partial} = \partial + \theta_z : \mathbb{C}((x))[z] \to \mathbb{C}((x))[z],$$

where $\partial(h(x)z^k)) = \partial(h(x))z^k = h'(x)z^k$ and where the map θ_u defined by

$$\theta_z(h(x)z^k) = h(x)\underline{\partial}(z^k) = kx^{-1}h(x)z^{k-1}$$

sends $\mathbb{C}((x))[z]_k$ to $\mathbb{C}((x))[z]_{k-1}$ for $k \geq 1$, and is 0 on $\mathbb{C}((x))$. This definition reflects, of course, the differentiation rule for the logarithm,

$$\partial(h(x)\log(x)^k) = h'(x)\log(x)^k + kx^{-1}h(x)\log(x)^{k-1}.$$

The image of $\underline{\partial}$ is the $\mathbb{C}$-subspace H of $\mathbb{C}((x))[z]$ of polynomials $\sum_{k=0}^m h_k(x)z^k$ satisfying the integrability condition

$$\partial(xh_{k-1}(x)) = kh_k(x)$$

for all $k \geq 0$. Integration on $\mathbb{C}((x))$ is now defined by $\int x^\rho = \frac{1}{\rho+1}x^{\rho+1}$ for $\rho \neq -1$ and $\int x^{-1} = u$ [we set the additive constants equal to 0]. It is a map $\int : \mathbb{C}((x)) \to \mathbb{C}((x))[z]$ which can be trivially extended to the subspace H of $\mathbb{C}((x))[z]$. Thus $\int : H \to \mathbb{C}((x))[z]$ is a right inverse to $\underline{\partial}$, say $\underline{\partial} \circ \int = \mathrm{Id}_H$. Arbitrary elements $\sum_{k=0}^m h_k(x)z^k \in \mathbb{C}((x))[z]$ cannot be integrated in general, so $\int$ does not extend to a map $\mathbb{C}((x))[z] \to \mathbb{C}((x))[z]$ inverse to $\underline{\partial}$.

Let L be an n-th order linear differential operator on $\mathbb{C}((x))$,

$$L = p_0\partial^n + p_1\partial^{n-1} + \ldots + p_{n-1}\partial + p_n,$$

with coefficients p_i in $\mathbb{C}((x))$. We extend L to the operator $\underline{L}$ on $\mathbb{C}((x))[z]$ defined by

$$\underline{L} = p_0\underline{\partial}^n + p_1\underline{\partial}^{n-1} + \ldots + p_{n-1}\underline{\partial} + p_n.$$

$$\underline{L}(h(x)z^k) = \partial h(x)z^k + kh(x)x^{-1}z^{k-1}.$$

This operator is now compatible with the substitution of z by $\log(x)$:

Proposition. *Let L be a differential operator on $\mathbb{C}((x))$ with extension $\underline{L}$ to $\mathbb{C}((x))[z]$ as defined above. Let $\rho \in \mathbb{C}$, $k \in \mathbb{N}$, and $h(x) \in \mathbb{C}[[x]]$ a formal power series. Then*

$$L(x^\rho h(x)\log(x)^k) = \underline{L}(x^\rho h(x)z^k)_{|z=\log(x)}.$$

Proof. This holds by definition of $\underline{\partial}$ and since $\log(x)' = x^{-1}$. ⟲

Corollary. *The solutions of a differential equation $Ly = 0$ in $\mathbb{C}((x))[\log(x)]$ are in bijection with the solutions of the associated equation $\underline{L}y = 0$ in $\mathbb{C}((x))[z]$.* ⟲

Example. The Euler equation $x^2y'' + 3xy' + 1 = 0$ with operator $L = x^2\partial^2 + 3x\partial + 1$ has indicial polynomial $\chi_L = \rho(\rho-1) + 3\rho + 1 = (\rho+1)^2$ with double root $\rho = -1$. It is immediately checked that $y_1 = x^{-1}$ and $y_2 = x^{-1}\log(x)$ are solutions of $Ly = 0$. The operator $\underline{L} = x^2\underline{\partial}^2 + 3x\underline{\partial} + 1$ therefore has, as it should be, solutions x^{-1} and $x^{-1}u$. Indeed, $\underline{\partial}(x^{-1}z) = x^{-2}(-z+1)$ and

$$\underline{\partial}^2(x^{-1}z) = \underline{\partial}(x^{-2}(-z+1)) = -2x^{-3}(-z+1) - x^{-3} = x^{-3}(2z-3).$$

Thus,

$$\underline{L}(x^{-1}z) = x^{-1}(2z-3) + 3x^{-1}(-z+1) + x^{-1}z = x^{-1}(2z-3-3z+3+z) = 0.$$

The proposition and its corollary guarantee that when we search for logarithmic solutions of a differential equation $Ly = 0$ we may study instead the differential equation $\underline{L}y = 0$ on $\mathbb{C}((x))[z]$, with $\underline{L}$ associated to L as above. This notational trick simplifies substantially the formulation of the problem.

The evaluation of the induced linear map $\underline{L} : \mathbb{C}((x))[z] \to \mathbb{C}((x))[z]$ on elements $\sum h_k z^k$ requires a multiple application of the product rule, since each $\underline{\partial}^j = (\partial + \theta_z)^j$ is a j-fold composition. We will see that there evolves a precise pattern which we will explore next. We first concentrate on Euler operators.

Examples. (1) Let $L = \partial$ and $\underline{L} = \underline{\partial}$. Then

$$\underline{\partial}(x^i z^k) = ix^{i-1}z^k + x^i x^{-1}kz^{k-1} = (iz+k)x^{i-1}z^{k-1}.$$

(2) Let $L = \partial^2$, $\underline{L} = \underline{\partial}^2$. Then

$$\begin{aligned}
\underline{\partial}^2(x^i z^k) &= \underline{\partial}(ix^{i-1}z^k + kx^i x^{-1}z^{k-1}) \\
&= \underline{\partial}(ix^{i-1}z^k + kx^{i-1}z^{k-1}) \\
&= i^{\underline{2}}x^{i-2}z^k + kix^{i-2}z^{k-1} + k(i-1)x^{i-2}z^{k-1} + k^{\underline{2}}x^{i-2}z^{k-2} \\
&= i^{\underline{2}}x^{i-2}z^k + (2i-1)kx^{i-2}z^{k-1} + k^{\underline{2}}x^{i-2}z^{k-2} \\
&= i^{\underline{2}}x^{i-2}z^k + (i^{\underline{2}})'kx^{i-2}z^{k-1} + \tfrac{1}{2}(i^{\underline{2}})''k^{\underline{2}}x^{i-2}z^{k-2},
\end{aligned}$$

where $(t^{\underline{2}})' = (t(t-1))' = 2t-1$ and $(t^{\underline{2}})'' = (t(t-1))'' = 2$ denote the first and second derivatives of $t^{\underline{2}}$ with respect to the variable t. This computation suggests a general formula for $\underline{\partial}^j$. Here it is.

Lemma 1. *For $j, \ell \in \mathbb{N}$ and $\rho \in \mathbb{C}$, denote by $(\rho^{\underline{j}})^{(\ell)}$ the evaluation at $t = \rho$ of the ℓ-th derivative $(t^{\underline{j}})^{(\ell)} := \partial_t^\ell(t^{\underline{j}})$ of the falling factorial $t^{\underline{j}} = t(t-1)\cdots(t-j+1)$. Then*

$$\begin{aligned}
\underline{\partial}^j(x^\rho z^k) &= \rho^{\underline{j}}x^{\rho-j}z^k + (\rho^{\underline{j}})'kx^{\rho-j}z^{k-1} + \ldots + \tfrac{1}{j!}(\rho^{\underline{j}})^{(j)}k^{\underline{j}}x^{\rho-j}z^{k-j}. \\
&= [\rho^{\underline{j}}z^j + (\rho^{\underline{j}})'kz^{j-1} + \ldots + \tfrac{1}{j!}(\rho^{\underline{j}})^{(j)}k^{\underline{j}}] \cdot x^{\rho-j}z^{k-j}.
\end{aligned}$$

Remark. A similar formula appears in [Mezzarobba, Prop. 4.14 and 4.16, p. 69, 70].

Proof. To prove the formula, use induction on j and the following identities. ⟲

Lemma 2. *The derivatives of the falling factorials satisfy, for $j \in \mathbb{N}$, the identities*

$$\begin{aligned}
&t^{\underline{j}} + (t^{\underline{j}})'(t-j) = (t^{\underline{j+1}})', \\
&(t^{\underline{j}})' + \tfrac{1}{2}(t^{\underline{j}})''(t-j) = \tfrac{1}{2}(t^{\underline{j+1}})'', \\
&\tfrac{1}{2}(t^{\underline{j}})'' + \tfrac{1}{6}(t^{\underline{j}})'''(t-j) = \tfrac{1}{6}(t^{\underline{j+1}})''', \\
&\ldots
\end{aligned}$$

$\frac{1}{(j-1)!}(t^{\underline{j}})^{(j-1)} + \frac{1}{j!}((t^{\underline{j}})^{(j)}(t-j) = \frac{1}{j!}(t^{\underline{j+1}})^{(j)},$

$\frac{1}{j!}((t^{\underline{j}})^{(j)} = \frac{1}{(j+1)!}((t^{\underline{j+1}})^{(j+1)}.$

The general formula for $\ell = 0, ..., j-1$ *is*

$$\frac{1}{\ell!}(t^{\underline{j}})^{(\ell)} + \frac{1}{(\ell+1)!}((t^{\underline{j}})^{(\ell+1)}(t-j) = \frac{1}{(\ell+1)!}(t^{\underline{j+1}})^{(\ell+1)}.$$

Proof. The first equation follows directly from the product rule, say

$$(t^{\underline{j+1}})' = (t^{\underline{j}}(t-j))' = (t^{\underline{j}})'(t-j) + t^{\underline{j}}.$$

The other identities are proven by successive differentiation of the first equation. For instance, deriving the first equation gives

$$(t^{\underline{j}})' + (t^{\underline{j}})''(t-j) + (t^{\underline{j}})' = (t^{\underline{j+1}})'',$$

which is just the second equation. Differentiation of the equation for the general formula gives

$$\frac{1}{\ell!}(t^{\underline{j}})^{(\ell+1)} + \frac{1}{(\ell+1)!}((t^{\underline{j}})^{(\ell+2)}(t-j) + \frac{1}{(\ell+1)!}((t^{\underline{j}})^{(\ell+1)} = \frac{1}{(\ell+1)!}(t^{\underline{j+1}})^{(\ell+2)}.$$

Then use $\frac{1}{\ell!} + \frac{1}{(\ell+1)!} = \frac{\ell+2}{(\ell+1)!}$ to get the next equation

$$\frac{1}{(\ell+1)!}(t^{\underline{j}})^{(\ell+1)} + \frac{1}{(\ell+2)!}((t^{\underline{j}})^{(\ell+2)}(t-j) = \frac{1}{(\ell+2)!}(t^{\underline{j+1}})^{(\ell+2)}.$$

This proves the claim. ◯

The next result, which follows from the above, will be the clue to understand why logarithms appear in the solutions of differential equations when the local exponents are multiple roots of the indicial polynomial.

Lemma 3. *Let* $L_0 = \sum_{i=0}^n c_i x^i \partial^i$ *be an Euler operator of shift* 0*, and let* $\underline{L}_0 = \sum_{i=0}^n c_i x^i \underline{\partial}^i$ *be the associated operator on* $\mathbb{C}((x))[z]$*. Denote by* $\chi(\rho) = \sum_{i=0}^n c_i \rho^{\underline{i}}$ *the indicial polynomial of* L_0*, and by* $\chi^{(j)}$ *its* j*-th derivative. Let* $\rho \in \mathbb{C}$ *and* $k \in \mathbb{N}$*. Then*

$$\underline{L}_0(x^\rho z^k) = x^\rho \cdot [\chi(\rho) z^k + \chi'(\rho) k z^{k-1} + \ldots + \tfrac{1}{n!}\chi^{(n)}(\rho) k^{\underline{n}} z^{k-n}].$$

Proof. This is a consequence of the formula for $\underline{\partial}^j(x^\rho z^k)$ in Lemma 1. ◯

Remark. Note that for $k < n$ (which will be the relevant case) no negative powers of z appear in the expansion of $\underline{L}(x^\rho z^k)$ because $k^{\underline{j}} = 0$ for $j > k$. In this case the formula reduces to

$$\underline{L}(x^\rho z^k) = x^\rho \cdot [\chi_L(\rho) z^k + \chi_L'(\rho) k z^{k-1} + \ldots + \tfrac{1}{(k-1)!}\chi_L^{(k-1)}(\rho) k^{\underline{k-1}} z + \tfrac{1}{k!}\chi_L^{(k)}(\rho) k!].$$

Proposition. *If* $\rho \in \mathbb{C}$ *is an* m*-fold root of the indicial polynomial* P_0 *of an Euler operator* L_0*, then* x^ρ*,* $x^\rho \log(x)$*,...,* $x^\rho \log(x)^{m-1}$ *are solutions of* $L_0 y = 0$*.*

Proof. Indeed, ρ being an m-fold root of χ signifies that $\chi^{(k)}(\rho) = 0$ for $k = 0, ..., m-1$, whence $\underline{L}_0(x^\rho z^k) = 0$. Substituting z by $\log(x)$ in this equation gives $L_0(x^\rho \log(x)^k) = 0$. ◯

Corollary. *Let* $L_0 = \sum_{i-j=\tau} c_{ij} x^i \partial^j \in \mathbb{C}((x))[\partial]$ *be an Euler operator with indicial polynomial* $\chi(\rho) = \sum_{j=0}^n c_{ij} \rho^{\underline{j}}$*. Let* $\rho_1, ..., \rho_q \in \mathbb{C}$ *be the distinct roots of* χ*, each with multiplicity* $m_1, ..., m_q$*, respectively. Then* x^{ρ_i}*,* $x^{\rho_i} \log(x)$*,...,* $x^{\rho_i} \log(x)^{m_i-1}$*,* $i = 1, ..., q$*, form a* $\mathbb{C}$*-basis of local solutions of* $L_0 y = 0$ *at* 0*.*

Proof. Clearly, these solutions are $\mathbb{C}$-linearly independent. As there cannot be more than $n = \operatorname{ord} L = \deg \chi = \sum_{i=1}^q m_i$ linearly independent solutions, they already form a $\mathbb{C}$-basis. ◯

Our next objective will be to “lift” the solutions of Euler equations to solutions of arbitary linear differential equations $Ly = 0$ by interpreting the operator L as an arbitrarily small perturbation of its initial form $L_0 = \mathrm{in}(L)$. This cannot work without some extra effort since whereas the initial form L_0 of an operator L sends monomials x^i in $\mathbb{C}((x))$ to monomials $x^{i+\tau}$, this is no longer true for $\underline{L}_0$, since already for $\underline{L}_0 = \underline{\partial}$ we have that $\underline{L}_0(x^i z^k) = (iz + k)x^{i-1}z^{k-1}$ is now a binomial in z. We may have to apply L_0^{-1} to each degree in z, say, $L^{-1} : \mathbb{C}((x))[z]_k \to \mathbb{C}((x))[z]_k$, instead of trying with $\underline{L}_0^{-1}$. But, of course, L_0^{-1} is no longer compatible with the substitution $z \to \log(x)$. So complications will have to be expected - but we will resolve them.

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NOTES PART IV: NORMAL FORM THEOREM WITHOUT LOGARITHMS

The situation we will consider in this section is as follows:

We denote by $\mathcal{O}$ the ring of germs at 0 of holomorphic functions on $\mathbb{C}$. It identifies with the ring $\mathbb{C}\{x\}$ of convergent power series. Its completion $\widehat{\mathcal{O}} = \mathbb{C}[[x]]$ consists of formal power series. We will treat both cases in parallel. Let $L = \sum_{j=0}^{n} \sum_{i=0}^{\infty} c_{ij} x^i \partial^j \in \mathcal{O}[\partial]$ be a linear differential operator with holomorphic coefficients. Decompose it into

$$L = L_0 + L_1 + L_2 + \cdots,$$

where L_i are Euler operators of increasing shifts $s_0 < s_1 < s_2 < \cdots$. Up to multiplication of L with a monomial in x we may and will assume that L_0 has shift 0, i.e., sends monomials x^k to $\chi(k)x^k$, where χ denotes the indicial polynomial of L_0 (or, of L) at 0. We call L_0 the initial form of L at 0. The roots ρ of χ in $\mathbb{C}$ are called the local exponents of L at 0, and their multiplicities are denoted by m_ρ.

We say that L has a *regular singularity* at 0 if L_0 is an operator of the same order as L. It is equivalent to say that the coefficient c_{nn} of L is non-zero, or that $\sum_\rho m_\rho = n$, or that $L = \sum_{j=0}^{n} a_j(x)\partial^j$ has quotients a_i/a_n which a pole of order at most $n - i$ at 0. Fuchs' original definition of regular singularities was formulated in terms of the solutions, claiming that $Ly = 0$ has a basis of *moderate* solutions at 0. In a later section we will make the meaning of "moderate" precise and prove the equivalence with the other definitions .

Examples. (1) The equation $x^k y' + y = 0$ has a regular singularity at 0 if and only if $k \leq 1$.

(2) The second order equation $x^k y'' + x^m y' + y = 0$ has a regular singularity at 0 if and only if $x^m/x^k = x^{m-k}$ has a pole of order $\leq 2 - 1 = 1$, and $1/x^k = x - k$ has a pole of order $\leq 2 - 0 = 2$ at 0. This is equivalent to $k \leq 2$ and $k \leq m + 1$. In particular, if $k = 2$, then $m \geq 1$.

(3) Consider now $x^2 y'' + 3x'y' + y - xy = 0$. The initial form at 0 is $L_0 = x^2\partial^2 + 3x\partial + 1$ with shift 0, while $L_1 = -x$, the multiplication with x, has shift $+1$. The indicial polynomial χ is $\rho(\rho-1)+3\rho+1 = (\rho+1)^2$, with root $\rho = -1$ of multiplicity $m_\rho = 2$. The associated Euler equation $L_0 y = 0$ has solutions $y_1 = x^{-1}$, $y_2 = x^{-1}\log(x)$. By the results of Fuchs-Thomé-Frobenius, the solutions of $Ly = 0$ are

$$y_1 = x^{-1}h_0(x), y_2 = x^{-1}h_1(x) + x^{-1}\log(x)h_0(x)$$

with holomorphic functions $h_0, h_1 \in \mathcal{O}$. We will prove this in the course of the classes in a modern and more conceptual language. Let us proceed step by step.

We start with the classical description of one specific local solution of a linear differential equation at a regular singular point, assuming an extra assumption on the involved local exponent ρ:

Theorem. [Fuchs, Thomé, Frobenius] *Let 0 be a regular singularity of an n-th order linear differential equation $Ly = 0$ with holomorphic coefficients, and let $\rho \in \mathbb{C}$ be a local exponent of L at 0. Assume that ρ is a maximal local exponent of L modulo $\mathbb{Z}$, i.e., that $\rho + k$ is not a local exponent for any integer $k \geq 1$. Then there exists a holomorphic function $h(x)$ in the neighborhood of 0 such that $y(x) = x^\rho \cdot h(x)$ is a solution of $Ly = 0$.*

We will establish this result as a corollary of the normal form theorem to be proven below. It goes as follows:

Theorem. (Normal form theorem vs1, maximal exponent, no logarithms) *Let $L \in \mathcal{O}[\partial]$ be an n-th order linear differential operator with holomorphic coefficients. Let $\rho \in \mathbb{C}$ be a maximal local exponent of L at 0 modulo $\mathbb{Z}$, i.e., such that $\rho + k$ is not a local exponent for any positive integer k. Denote by L_0 the initial form of L at 0, and assume that L_0 has shift 0. Set $\mathcal{F} = x^\rho \mathcal{O}$ and $\widehat{\mathcal{F}} = x^\rho \widehat{\mathcal{O}}$ and write also L and $\widehat{L}$ for the linear maps on $\mathcal{F}$ and $\widehat{\mathcal{F}}$ induced by L. There exists a linear automorphism*

$$\widehat{u} : \widehat{\mathcal{F}} \to \widehat{\mathcal{F}}$$

such that on $\widehat{\mathcal{F}}$

$$\widehat{L} \circ \widehat{u}^{-1} = \widehat{L}_0.$$

If 0 is a regular singular point of L, then $\widehat{u}$ restricts to a linear automorphism

$$u : \mathcal{F} \to \mathcal{F}$$

such that on $\mathcal{F}$

$$L \circ u^{-1} = L_0.$$

This result justifies the wording that L_0 is a *normal form* of L on $\mathcal{F}$. From this we immediately obtain

Corollary. *Let $y_1 = x^\rho$ be the first solution of the associated Euler equation $L_0 y = 0$. Then $u^{-1}(y_1) = u^{-1}(x^\rho)$ is a solution of $Ly = 0$.* $\circlearrowleft$

Remarks. (a) A suitable u is given as

$$u = \mathrm{Id}_{x^\rho \cdot \mathbb{C}\{x\}} - S \circ T,$$

where $S = L_{0|\mathcal{H}}^{-1}$ is the inverse of the restriction of L_0 to a direct complement $\mathcal{H}$ of its kernel in $\mathcal{F}$, and where $T = L_0 - L$ is the negative of the tail of L. Its inverse $v = u^{-1}$ is given as the geometric (or: von Neumann) series $v = \mathrm{Id}_\mathcal{F} + \sum_{k=1}^\infty (S \circ T)^k$, see the proof.

(b) In case that ρ has multiplicity $m_\rho > 1$, the other solutions $y_i = x^\rho \log(x)^{i-1}$ of $L_0 y = 0$ can also be lifted to solutions of $Ly = 0$, but this requires to introduce logarithms in the function space $\mathcal{F}$. See below part V.

(c) The regularity of the singularity of L is used only for the convergence part, i.e., that $\widehat{u}$ sends $\mathcal{F}$ into $\mathcal{F}$. Later on, for constructing a whole basis of solutions, it will be used again so as to have sufficiently local exponents, namely such that their multiplicities $\sum m_\rho = n$ sum up to n.

(d) The maximality of ρ with respect to $\mathbb{Z}$ among the local exponents of L is crucial. If this is not assumed, more complicated function spaces $\mathcal{F}$ have to be considered, both for the normal form theorem and the description of the solutions as in the corollary.

(e) The part for formal power series works for any field of characteristic 0. The case of positive characteristic is much more complicated and has been developed and proven recently by Florian Fürnsinn from the University of Vienna.

Here is a (preliminary) generalization of the normal form theorem from above, letting now L act on a space $\mathcal{F}$ containing powers of logarithms, codified again through a new variable z. The precise statement and the proof will be given in part V of the notes.

Theorem. (Normal form theorem vs2, maximal exponent, with logarithms) *Let $L \in \mathcal{O}[\partial]$ be an n-th order linear differential operator with holomorphic coefficients. Let $\rho \in \mathbb{C}$ be a maximal local exponent of L at 0 modulo $\mathbb{Z}$, i.e., such that $\rho + k$ is not a local exponent for any positive integer k. Let $m = m_\rho \geq 1$ be its multiplicity. Denote by L_0 the initial form of L at 0, and assume that L_0 has shift 0. Set $\mathcal{F} = x^\rho \mathcal{O}[z]_{<m}$ and $\widehat{\mathcal{F}} = x^\rho \widehat{\mathcal{O}}[z]_{<m}$, where $< m$ stands for polynomials in z of degree $< m$. Denote by $\underline{\partial}$ the extension of ∂ to $\mathcal{O}[z]$ defined by $\underline{\partial} x = 1$, $\underline{\partial} z = x^{-1}$, and write accordingly $\underline{L} \in \mathcal{O}[\underline{\partial}]$ for the induced operator. There exists a linear automorphism $\widehat{u} : \widehat{\mathcal{F}} \to \widehat{\mathcal{F}}$ such that on $\widehat{\mathcal{F}}$*

$$\underline{L} \circ \widehat{u}^{-1} = \underline{L}_0.$$

If 0 is a regular singular point of L, then $\widehat{u}$ restricts to a linear automorphism $u : \mathcal{F} \to \mathcal{F}$ such that on $\mathcal{F}$

$$\underline{L} \circ u^{-1} = \underline{L}_0.$$

Again, we immediately obtain

Corollary. *Let $y_1 = x^\rho, ..., y_{m_\rho} = x^\rho \log(x)^{m_\rho - 1}$ be the solutions of the Euler equation $L_0 y = 0$. Then $u^{-1}(y_1) = u^{-1}(x^\rho), ..., u^{-1}(y_{m_\rho}) = u^{-1}(x^\rho \log(x)^{m_\rho - 1})$ are solutions of $Ly = 0$.* ○

Remarks. (a) To get a basis of solutions of $Ly = 0$ one has to vary the local exponents ρ. But there might occur two obstructions: First, some local exponents may not be maximal, and then extra caution has to be taken; and, secondly, there the sum of the multiplicities m_ρ may be strictly less than n. In this case, as seen above, the singularity is not regular. There still exists a basis of solutions in a suitable function space. It will involve exponentials $\exp(P(1/x^q))$ of rational functions, P a polynomial, $q \in \mathbb{N}$ an integer. This is a classical theorem of Fabry from 1885. Nicholas Merkl from the University of Vienna is currently preparing a modern version of it along the lines of the normal form theorem.

(b) The special shape of the solutions $u^{-1}(y_i)$ of $Ly = 0$ as indicated in the theorem of Fuchs-Thomé-Frobenius above follows from the explicit description of a suitable automorphism u.

Proof. (a) We only prove here the normal form theorem in case where L acts on $\mathcal{F} = x^\rho \widehat{\mathcal{O}}$, respectively, $\mathcal{F} = x^\rho \mathcal{O}$. This will already reveal the technique and the various arguments. The case of $\underline{L}$ acting on $\mathcal{F} = x^\rho \mathcal{O}[z]$ goes along the same lines, and we will indicate the places where modifications have to be applied.

Write $L = L_0 - T$ with $-T = L_1 + L_2 + \cdots$ the tail of L. As T has positive shift (recall that L_0 is assumed to have shift 0), we have

$$T(x^\rho \mathcal{O}) \subset x^{\rho+1} \mathcal{O}.$$

This will be used in a moment. As L_0 annihilates x^ρ, we also get $L_0(x^\rho \mathcal{O}) \subset x^{\rho+1} \mathcal{O}$. But now, as ρ is maximal modulo $\mathbb{Z}$ among the local exponents of L, we know that $L_0(x^{\rho+k}) \neq 0$ for all $k \geq 1$. This implies that $L_0(\mathcal{F}) = x\mathcal{F}$. It hence contains the image of T. This is crucial for the argument to follow, and it also holds for $\underline{L}$ and $\mathcal{F} = x^\rho \mathcal{O}[z]$, as one checks with a little patience.

Now, as $L_0 : \mathcal{F} \to x\mathcal{F}$ is surjective, it induces a linear isomorphism when restricted to a direct complement $\mathcal{H}$ of the kernel $\mathrm{Ker}(L_0)$ of L_0 in $\mathcal{F}$,

$$L_{0|\mathcal{H}} : \mathcal{H} \to x\mathcal{F}.$$

Denote by $S = L_{0|\mathcal{H}}^{-1}$ its inverse,

$$S : x\mathcal{F} \to \mathcal{H}.$$

(b) At this point the proof splits into two case, the case of formal power series and the one with convergent series. Let us do first the formal case, $\widehat{O} = \mathbb{C}[[x]]$, and write $\widehat{u} : \widehat{\mathcal{H}} \to x\widehat{\mathcal{F}}$ for the map defined above (we still write L, S and T without "$\widehat{\ }$").

We claim that

$$\widehat{v} := \mathrm{Id}_{\widehat{\mathcal{F}}} + \textstyle\sum_{k=1}^{\infty} (S \circ T)^k : \mathcal{F} \to \mathcal{F}$$

is well defined and an inverse to $\widehat{u}$. To see this, juste recall that T, when applied to a power seres, increases its order at least by 1. And S preserves the order, since L_0 has shift 0. Actually, one may choose for S the map (the "integration operator") defined by

$$S(x^{\rho+k}) = \frac{1}{\chi(\rho+k)} \cdot x^{\rho+k}.$$

So $S \circ T$ maps $x^m\mathcal{F}$ into $x^{m+1}\mathcal{F}$. But as $\mathbb{C}[[x]]$ is complete with respect to the x-adic topology (with neighborhoods of 0 given by the powers $(x)^m$ of the maximal ideal (x)), we can conclude that $\widehat{v}$ defines indeed a map from $\mathcal{F}$ to $\mathcal{F}$. And clearly, $\widehat{v}$ is then an inverse to $\widehat{u}$, all maps being linear.

It remains to prove that $L_{0|\mathcal{F}} \circ u = L_{|\mathcal{F}}$, where we write subscripts to emphasize that we mean the linear maps on $\mathcal{F}$ and not the abstract operators. It is also helpful to convince oneself that in the equations below all computations are valid transformations. The proof of the equality is now easy (and nice). Namely, we have

$$\begin{aligned}
L_{0|\mathcal{F}} \circ u &= L_{0|\mathcal{F}} \circ (\mathrm{Id}_{\mathcal{F}} - S \circ T) \\
&= L_{0|\mathcal{F}} \circ (\mathrm{Id}_{\mathcal{F}} - S \circ (L_0 - L)_{|\mathcal{F}}) \\
&= L_{0|\mathcal{F}} \circ (\mathrm{Id}_{\mathcal{F}} - S \circ L_{0|\mathcal{F}} + S \circ L_{|\mathcal{F}}) \\
&= L_{0|\mathcal{F}} - L_{0|\mathcal{F}} \circ S \circ L_{0|\mathcal{F}} + L_{0|\mathcal{F}} \circ S \circ L_{|\mathcal{F}} \\
&= L_{0|\mathcal{F}} \circ S \circ L_{|\mathcal{F}} \\
&= L_{|\mathcal{F}},
\end{aligned}$$

using twice that S is an inverse to $L_{0|\mathcal{H}}$ and hence $L_{0|\mathcal{F}} \circ S = \mathrm{Id}_{x\mathcal{F}}$. And recall that L maps $\mathcal{F}$ into $x\mathcal{F}$, so all compositions are well defined. This concludes the proof of the theorem for formal power series.

It is not hard to see that the same reasoning applies to the extension $\mathcal{F} = x^\rho\mathcal{O}[z]$ and the associated linear map $\underline{L}_{|\mathcal{F}}$. We leave this as an exercise.

(c) As for the convergent case, one has to look a bit carefully what the maps T and S do on power series with prescribed radius of convergence. Let $s > 0$ be a small real number and denote by $\mathcal{O}_s$ the ring of power series $h = \sum_{k=0}^{\infty} a_k x^k$ such that $|h|_s := \sum_{k=0}^{\infty} |a_k| s^k < \infty$. This is a Banach space, and $|-|_s$ is a norm on it. We thus get the induced Banach space $\mathcal{F}_s$ and linear maps $\mathcal{F}_s \to \mathcal{F}_s$ can be equipped with the operator norm (in the sense of functional analysis), denoted by $||-||_s$. We may choose $s > 0$ sufficiently small such that $L : \mathcal{F}_s \to \mathcal{F}_s$ is well defined (recall that $L \in \mathcal{O}[\partial]$ has finitely many convergent coefficients, hence belongs to $\mathcal{O}_s[\partial]$ for s small).

We show that $||(S\circ T)||_s<1$. This will imply the convergence of the sum $\sum_{k=0}^\infty (S\circ T)^k$ defining v as a map on $\mathcal{F}_s$. To this end, we show that there is a constant $0<C<1$ such that

$$|S(T(x^\rho h))|_s\leq C\cdot |x^\rho h|_s$$

for all $h\in\mathcal{O}_s$. The formulas are slightly complicated. For $h=\sum_{k=0}^\infty a_kx^k\in\mathcal{O}$ we have

$$T(x^\rho h)=-\sum_{i-j>0}\sum_{k=0}^\infty (\rho+k)^{\underline{j}}\, c_{ij}a_kx^{\rho+k+i-j},$$

and

$$S(T(x^\rho h))=-\sum_{i-j>0}\sum_{k=0}^\infty \frac{(\rho+k)^{\underline{j}}}{\chi(\rho+k+i-j)}\, c_{ij}a_kx^{\rho+k+i-j}.$$

As $i-j>0$ and $k\geq 0$, no $\rho+k+i-j$ appearing in the denominator is a root of the indicial polynomial χ. Hence the ratio

$$\frac{(\rho+k)^{\underline{j}}}{\chi(\rho+k+i-j)}=\frac{(\rho+k)^{\underline{j}}}{\sum_{\ell-m=0}c_{\ell m}(\rho+k+i-j)^{\underline{m}}}=\frac{(\rho+k)^{\underline{j}}}{\sum_{m=0}^n c_{mm}(\rho+k+i-j)^{\underline{m}}}$$

is well defined. But, as the singularity of L is regular, the order of L_0 is n and hence $c_{nn}\neq 0$. This implies that $(\rho+k+i-j)^{\underline{n}}$ appears in the denominator with non-zero coefficient. It is at that place where the regularity of the singularity is used in the proof. Note that in the ratio some i could be less than j and hence the respective $\rho+k+i-j$ would be smaller than the $\rho+k$ in the numerator. But the bound $j\leq n$ for j nevertheless ensures that the ratio remains bounded, say $\leq c$, in absolute value as k tends to ∞. Taking norms on both sides of the above equality for $S(T(x^\rho h))$ yields, for $s\leq 1$, the estimate

$$|S(T(x^\rho h))|_s\leq c\sum_{i-j>0}\sum_{k=0}^\infty |c_{ij}||a_k|s^{\rho+k+i-j}=c\sum_{i-j>0}|c_{ij}|s^{i-(j)}\sum_{k=0}^\infty |a_k|s^{\rho+k}.$$

But by assumption the coefficients $\sum_{i=0}^\infty c_{ij}x^i$ of L belong to $\mathcal{O}_s$ for all $j=0,...,n$. This implies in particular $\sum_{i>j}^\infty c_{ij}x^i\in\mathcal{O}_s$ and then, after division by x^{j+1} and since $i\geq j+1$, that

$$\textstyle\sum_{i>j}^\infty c_{ij}x^{i-(j+1)}\in\mathcal{O}_s.$$

We get that

$$\textstyle\sum_{i-j>0}|c_{ij}|s^{i-(j)}=s\cdot\sum_{i-j>0}|c_{ij}|s^{i-(j+1)}\leq c's$$

for some $c'>0$ independent of s. This inequality allows us to bound $|S(T(x^\rho h))|_s$ from above by

$$|S(T(x^\rho h))|_s\leq cc's\sum_{k=0}^\infty |a_k|s^{\rho+k}=cc's|x^\rho h|_s.$$

Take now $0<s_0$ sufficiently small with $s_0<\frac{1}{cc'}$, and get a constant $0<C<1$ such that for $0<s\leq s_0$ one has

$$|S(T(x^\rho h))|_s\leq C\cdot|x^\rho h|_s.$$

This shows that $||(S\circ T)_s||<1$ holds on F_s for $0<s\leq s_0$ as required. The convergence proof is completed. ⥀

Remark. The convergence proof extends to $\mathcal{F}=x^\rho\mathcal{O}[z]_{<m}$ (polynomials in z of degree $<m$) since this is a finite free module over $\mathcal{O}$. As the action of $\underline{L}$ on this extended $\mathcal{F}$ does not increase the degree in z, the restriction to this module is justified.

NOTES PART V: NORMAL FORM THEOREM WITH LOGARITHMS

Let us recall from last time the second, more general version of the normal form theorem where the local exponent ρ has multiplicity $m = m_\rho \geq 1$ and where we considered the operator L as acting on the enlarged function space $\mathcal{F} = x^\rho \mathcal{O}[z]$. In fact, we can (and will) even restrict to the smaller space $\mathcal{F} = x^\rho \mathcal{O}[z]_{<m}$ of polynomials in z of degree $< m$. Normalizing the action of the operator L on this space will be sufficient and perfectly suited to construct m solutions $y_i = x^\rho \log(x)^i h_i(x)$ of $Ly = 0$, with $0 \leq i < m$ and $h_i \in \mathcal{O}$ holomorphic. We will still have to assume that ρ is maximal with respect to $\mathbb{Z}$. The case where ρ is no longer maximal requires extra constructions and will be treated next time.

Theorem. (Normal form theorem vs2, maximal exponent, with logarithms) *Let $L = \sum_{j=0}^n p_j(x)\partial^j \in \mathcal{O}[\partial]$ be an n-th order linear differential operator with holomorphic coefficients p_j in $\mathcal{O}$. Let $\rho \in \mathbb{C}$ be a maximal local exponent of L at 0 modulo $\mathbb{Z}$, i.e., $\rho+k$ is not a local exponent for any positive integer k. Let $m = m_\rho \geq 1$ be its multiplicity as a root of the indicial polynomial χ of L. Denote by L_0 the initial form of L at 0, and assume that L_0 has shift 0. Set $\mathcal{F} = x^\rho \mathcal{O}[z]_{<m}$ and $\widehat{\mathcal{F}} = x^\rho \widehat{\mathcal{O}}[z]_{<m}$. Denote by $\underline{\partial}$ the extension of ∂ to $\mathcal{O}[z]$ defined by $\underline{\partial} x = 1$, $\underline{\partial} z = x^{-1}$, and write accordingly $\underline{L} = \sum_{j=0}^n p_j \underline{\partial}^j \in \mathcal{O}[\underline{\partial}]$ for the induced operator. There exists a linear automorphism $\widehat{u} : \widehat{\mathcal{F}} \to \widehat{\mathcal{F}}$ such that the linear maps$\underline{L}$ and $\underline{L}_0$ on $\widehat{\mathcal{F}}$ induced by L and L_0 satisfy*

$$\underline{L} \circ \widehat{u}^{-1} = \underline{L}_0.$$

Moreover, if 0 is a regular singular point of L, then $\widehat{u}$ restricts to a linear automorphism $u : \mathcal{F} \to \mathcal{F}$ such that the linear maps on $\mathcal{F}$ induced by $\underline{L}$ and $\underline{L}_0$ satisfy

$$\underline{L} \circ u^{-1} = \underline{L}_0.$$

Remarks. (a) The automorphism $\widehat{u}$ is again of the form $\widehat{u} = \mathrm{Id}_{\widehat{\mathcal{F}}} - S \circ T$ with $T = L_0 - L$ and S the inverse of the restriction $L_{0|\widehat{\mathcal{H}}}$ of L_0 to a direct complement $\widehat{\mathcal{H}}$ of its kernel in $\widehat{\mathcal{F}}$ as in the first version of the normal form theorem. Accordingly, u has the form $u = \mathrm{Id}_{\mathcal{F}} - S \circ T$.

(b) We do not allow L to have coefficients depending also on the variable z, i.e., lying in $\mathcal{O}[z]$. This would correspond to differential equations whose coefficients involve powers of logarithms. It is not clear whether this case would have interesting applications.

(c) The convergence part of the theorem requires again that 0 is a regular singularity of L. The proof is analogous to the case without logarithms, using the same estimates.

Before proving the theorem let us state immediately its central output about the solutions of $Ly = 0$:

Corollary. *Let $y_1 = x^\rho, ..., y_m = x^\rho \log(x)^{m-1}$ be the solutions of the Euler equation $L_0 y = 0$. Then $u^{-1}(y_1) = u^{-1}(x^\rho), ..., u^{-1}(y_m) = u^{-1}(x^\rho \log(x)^{m-1})$ are solutions of $Ly = 0$. More explicitly, these solutions are of the form, for $1 \leq i \leq m$,*

$$y_1(x) = x^\rho h_1(x),$$

$$y_2(x) = x^\rho [h_2(x) + h_1(x)\log(x)],$$

$$y_i(x) = x^\rho [h_i(x) + h_{i-1}(x)\log(x) + \ldots + h_1(x)\log(x)^{i-1}],$$

with $h_1, ..., h_m$ formal power series in $\widehat{O}$, respectively, holomorphic functions in $\mathcal{O}$.

Remark. The special shape of the solutions $y_i(x)$ stems from the explicit description of the normalizing automorphism u as given above, see the proof of the theorem.

Examples. The proof of the theorem will use the three lemmata 1, 2, 3 of part III (the section on Euler operators) describing the extensions $\underline{\partial}^j$ and $\underline{L}$ of derivations and differential operators to $x^\rho\mathcal{O}[z]$. To get a more concrete hold on these, let us consider two examples.

(1) Let $E = x^2\partial^2 - 3x\partial + 3$ be an Euler operator with indicial polynomial $\chi(t) = (t+1)^2$ and local exponent $\rho = -1$ of multiplicity $m = 2$. Let it act on $x^{-1}\mathcal{O}[z]_{<2}$. Then

$$\underline{E}(x^k z^i) = x^k[(k+1)^2 z^i + 2(k+1)iz^{i-1} + 2i(i-1)z^{i-2}].$$

We get $\mathrm{Ker}(E) = \mathbb{C}x^{-1} \oplus \mathbb{C}x^{-1}z$, and $\mathrm{Im}(E) = xx^{-1}\mathcal{O}[z]_{<2} = \mathcal{O}[z]_{<2}$.

(2) Let $E = x^3\partial^3 - 4x^2\partial^2 + 9x\partial - 9$ be an Euler operator with indicial polynomial $\chi(t) = (t-1)(t-3)^2$ and local exponents $\rho = 3$ of multiplicity $m = 2$ and $\sigma = 1$ of multiplicity 1. Let it act on $x\mathcal{O}[z]_{<2}$. Then

$$\underline{E}(x^k z^i) = x^k[(k-1)(k-3)^2 z^i + (3k-5)(k-1)iz^{i-1} + (6k-14)i^{\underline{2}}z^{i-2} + 6i^{\underline{3}}z^{i-3}].$$

The kernel is $\mathrm{Ker}(E) = \mathbb{C}x \oplus \mathbb{C}x^3 \oplus \mathbb{C}x^3 z$. Determine the image $\mathrm{Im}(E)$!

(3) Let finally $E = x^2\partial^2 - x\partial$ be with $\chi(t) = t(t-2)$ and local exponents $\rho = 2$ and $\sigma = 0$, both of multiplicity 1. Let it act on $\mathcal{F} = \mathcal{O} + x^2\mathcal{O} = \mathcal{O}$ since no logarithms are to be expected. Then

$$\underline{E}(x^k) = k(k-2)x^k$$

and hence $\mathrm{Ker}(E) = \mathbb{C} \oplus \mathbb{C}x^2$. The image is $\mathrm{Im}(E) = \mathbb{C}x + \mathcal{O}x^3 \subset \mathcal{O}$, which is now strictly contained in $x\mathcal{F} = x\mathcal{O}$. The "gap" occurs at x^2, and this will cause serious problems when trying to apply the arguments of the proof of the normal form theorem - recall that it relied heavily on the equality $\underline{L_0}(\mathcal{F}) = x\mathcal{F}$, and this fails in the present example. The reason is that there is *resonance* between the two local exponents, say, $\rho - \sigma \in \mathbb{Z}$. We will show in part VI of the notes how to overcome this problem.

Proof. Recall from Lemma 1, part III, the formula

$$\underline{\partial}^j = \partial^j + (\partial^j)'\partial_z + \tfrac{1}{2}(\partial^j)''\partial_z^2 + \ldots + \tfrac{1}{\ell!}(\partial^j)^{(\ell)}\partial_z^\ell + \ldots + \tfrac{1}{j!}(\partial^j)^{(j)}\partial_z^j,$$

where the derivatives $(\partial^j)^{(\ell)}$ are defined on $\mathcal{O}$ by $(\partial^j)^{(\ell)}(x^t) = (t^{\underline{j}})^{(\ell)}x^{t-j}$ while leaving z invariant. For an operator $L = \sum_{j=0}^n p_j\partial^j \in \mathcal{O}[\partial] \in \mathcal{O}[\partial]$ define accordingly its ℓ-th derivative as

$$L^{(\ell)} = \textstyle\sum_{j=0}^n p_j(\partial^j)^{(\ell)},$$

acting again on $\mathcal{O}$ while leaving z invariant. More explicitly,

$$(L^{(\ell)}\partial_z^\ell)(x^k z^i) = L^{(\ell)}(x^k)\cdot\partial_z^\ell(z^i) = (k^{\underline{j}})^{(\ell)}i^{\underline{\ell}}\cdot x^{k-j}z^{i-\ell}.$$

Then the preceding formula extends for the action $\underline{L}$ of L on $x^\rho\mathcal{O}[z]$ by linearity to

Lemma 4. *In the above situation, one has*

$$\underline{L} = L + L'\partial_z + \tfrac{1}{2}L''\partial_z^2 + \ldots + \tfrac{1}{\ell!}L^{(\ell)}\partial_z^\ell + \ldots + \tfrac{1}{n!}L^{(n)}\partial_z^n.$$

We will call this decomposition the *Taylor expansion* of $\underline{L}$ on $x^\rho\mathcal{O}[z]$.

Let us now turn to the actual proof of the normal form theorem with logarithms. The formula in Lemma 4 applies in particular to the initial form L_0 of L at 0. The key step is then, taking as function space on which

L and L_0 act the space $\mathcal{F} = x^\rho \mathcal{O}[z]_{<m}$, where $m = m_\rho$ is again the multiplicity of the local exponent ρ of L, the following

Claim. *The image of* $\underline{L}_0$ *acting on* $\mathcal{F}$ *is* $x\mathcal{F}$,

$$\underline{L}_0(\mathcal{F}) = x\mathcal{F}.$$

Proof. The inclusion $\underline{L}_0(\mathcal{F}) \subset x\mathcal{F}$ is straightforward. Indeed, if $i < m$ then $i^{\underline{\ell}} = 0$ for $\ell \geq m \geq i+1$. Therefore, the formula of Lemma 3,

$$\underline{L}_0(x^\rho z^i) = x^\rho \cdot [\chi(\rho) z^i + \chi'(\rho) i z^{i-1} + \ldots + \tfrac{1}{n!}\chi^{(n)}(\rho) i^{\underline{n}} z^{i-n}],$$

reduces to

$$\underline{L}_0(x^\rho z^i) = x^\rho \cdot [\chi(\rho) z^i + \chi'(\rho) i z^{i-1} + \ldots + \tfrac{1}{(m-1)!}\chi^{(m-1)}(\rho) i^{\underline{m-1}} z^{i-m+1}].$$

This implies that $\underline{L}_0(x^\rho z^i) = 0$ for $i < m$, say $\underline{L}_0(x^\rho \mathbb{C}[z]_{<m} = 0$. Thus $\underline{L}_0(\mathcal{F}) \subset x\mathcal{F}$.

For the converse inclusion $\underline{L}_0(\mathcal{F}) \supset x\mathcal{F}$ we have to show that $x^{\rho+k} z^i \in \underline{L}_0(\mathcal{F})$ for all $k \geq 1$ and all $0 \leq i < m$. This is immediate if $i = 0$: then

$$\underline{L}_0(x^{\rho+k}) = L_0(x^{\rho+k}) = \chi(\rho+k) x^{\rho+k}$$

and $\chi(\rho+k) \neq 0$ since ρ is maximal modulo $\mathbb{Z}$. So $x^{\rho+k} \in \underline{L}_0(\mathcal{F})$ for all $k \geq 1$. Let now $i > 0$. We apply induction on i. By Lemma 4 we know that

$$\underline{L}_0(x^{\rho+k} z^i) = L_0(x^{\rho+k} z^i) + \sum_{\ell=1}^n \frac{1}{\ell!} L_0^{(\ell)} \partial_z^\ell (x^{\rho+k} z^i).$$

In terms of the derivatives of the indicial polynomial χ this reads as

$$\underline{L}_0(x^{\rho+k} z^i) = \chi(\rho+k) x^{\rho+k} z^i + \sum_{\ell=1}^n \frac{1}{\ell!} \chi^{(\ell)}(\rho+k) \frac{1}{\ell!} i^{\underline{\ell}} \cdot x^{\rho+k} z^{i-\ell}.$$

The first summand is non-zero as before, and the polynomial in z defined by the sum of the second summand has degree $< i$, hence belongs by induction on i to $\underline{L}_0(\mathcal{F})$. Therefore $x^{\rho+k} z^i \in \underline{L}_0(\mathcal{F})$ and the converse inclusion $\underline{L}_0(\mathcal{F}) \supset x\mathcal{F}$ is shown.

From this point on, the proof follows exactly the proof of the normal form theorem from part IV of the notes. The only thing to remark is that, for the convergence proof, one uses the fact that $\mathcal{F} = x^\rho \mathcal{O}[z]_{<m}$ is a *finite* free $\mathcal{O}$-module and so the Banach space argument applies again. ○

NOTES PART VI: NORMAL FORM THEOREM, GENERAL CASE

Up to now we have always considered a local exponent ρ of $L \in \mathcal{O}[6]$ which was maximal modulo $\mathbb{Z}$, and we proved for this case the normal form theorem on the function space $\mathcal{F} = x^\rho \mathcal{O}[z]_{<m}$, where m is the multiplicity of ρ as a root of the indicial polynomial χ of L. We will now treat the general case of arbitrary local exponents. It turns out that local exponents which differ by an integer pose extra problems. We will call the occurrence of integer differences $resonance$.

To motivate our constructions, let us recall that the key step in the proof of the normal form theorem was to determine the image $\underline{L}_0(\mathcal{F})$ of the initial form L_0 of L and to prove that actually $L_0(\mathcal{F}) = x\mathcal{F}$. From this follows that the tail $T = L_0 - L$ of L sends $\mathcal{F}$ into the image of L_0, which is the critical property used to construct the normalizing automorphism u.

Let us illustrate first that a suitable definition of $\mathcal{F}$ is so obvious in case of resonance.

$Example.$ Let $L_0 = x^2\partial^2 - x\partial$ be an Euler operator of shift 0 and with indicial polynomial $\chi(t) = t(t-2)$. The local exponents are $\sigma = 0$ and $\rho = 2$, both of multiplicity 1. As both σ and ρ are simple roots of χ, one may expect that we can dispense of logarithms. A natural candidate for $\mathcal{F}$ seems to be $\mathcal{F} = x^\sigma\mathcal{O} + x^\rho\mathcal{O} = \mathcal{O} + x^2\mathcal{O} = \mathcal{O}$. Let us compute its image under L_0. We get

$$L_0(\mathcal{O}) = \mathbb{C}x + x^3\mathcal{O} \subsetneq x\mathcal{F} = x\mathcal{O}.$$

So the image is strictly contained in $x\mathcal{F}$. If you now take $L = L_0 - x^2\partial$ with $T = x^2\partial$, we see that $T(x) = x^2$ is not contained in $x\mathcal{F}$. So the construction of the automorphism u breaks down.

To remedy this failure, let us introduce logarithms in $\mathcal{F}$. We will describe two options to do this. The first one turns out to be unsuccessful, while the second will work.

$Attempt$ 1. Take

$$\mathcal{F} = (x^\sigma\mathcal{O} + x^\rho\mathcal{O})[z]_{<2} = \mathcal{O}[z]_{<2} = \mathcal{O} \oplus \mathcal{O}z.$$

This looks like a reasonable choice. Let us write $\underline{L}_0 = x^2\underline{\partial}^2 - x\underline{\partial}$ for the action of the extension of L_0 to $\mathcal{F}$. We get

$$\underline{L}_0(\mathcal{F}) = \mathbb{C}x + x^3\mathcal{O} + \underline{L}_0(\mathcal{O}z).$$

To compute the last summand, recall from Lemma 3 in part III that

$$\underline{L}_0(x^k z) = x^k[\chi(k)z + \chi'(k)].$$

As $\chi(t) = t(t-2)$ and $\chi'(t) = 2(t-1)$ this gives

$k = 0$: $\quad \underline{L}_0(z) = 0z + 2(0-1) = -2;$

$k = 1$: $\quad \underline{L}_0(xz) = -xz + 0 = -xz;$

$k = 2$: $\quad \underline{L}_0(x^2z) = 0x^2z + 2x^2 = 2x^2;$

$k = 3$: $\quad \underline{L}_0(x^3z) = 3x^3z + 4x^3.$

Already the first case $k = 0$ shows that $\underline{L}_0(\mathcal{F}) \not\subset x\mathcal{F}$. So this choice of $\mathcal{F}$ is not appropriate.

Attempt 2. Let us now take

$$\mathcal{F} = x^\sigma \mathcal{O}[z]_{<1} + x^\rho \mathcal{O}[z]_{<2} = \mathcal{O} + x^2\mathcal{O} + x^2\mathcal{O}z = \mathcal{O} \oplus x^2\mathcal{O}z.$$

We have $\underline{L}_0(\mathcal{F}) = \underline{L}_0(\mathcal{O}) + \underline{L}_0(x^2\mathcal{O}z)$ with $\underline{L}_0(\mathcal{O}) = L_0(\mathcal{O}) = \mathbb{C}x + x^3\mathcal{O}$ as before. As for $\underline{L}_0(x^2\mathcal{O}z)$, use again Lemma 3 to compute $\underline{L}_0(x^k z)$. We get as before

$$k = 2: \qquad \underline{L}_0(x^2 z) = 2x^2;$$

$$k = 3: \qquad \underline{L}_0(x^3 z) = 3x^3 z + 4x^3.$$

$$k = 4: \qquad \underline{L}_0(x^3 z) = 8x^4 z + 6x^4.$$

This implies

$$\underline{L}_0(\mathcal{F}) = \underline{L}_0(\mathcal{O} \oplus x^3\mathcal{O}z) = \mathbb{C}x + x^3\mathcal{O} + \mathbb{C}x^2 + x^3\mathcal{O}z = x(\mathcal{O} \oplus x^2\mathcal{O}z) = x\mathcal{F}.$$

That is precisely what we want - and it gives us a hint of how to define $\mathcal{F}$ in general.

Lemma 5. *Let $E \in \mathcal{O}[z]$ be an Euler operator with shift 0. Let Ω be a set of local exponents of E with integer differences, ordered increasingly,*

$$\rho_1 < \rho_2 < \ldots < \rho_r,$$

meaning that $\rho_{k+1} - \rho_k \in \mathbb{N}_{>0}$. Let m_k be the multiplicity of ρ_k. Set

$$\mathcal{F} = x^{\rho_1}\mathcal{O}[z]_{<m_1} + x^{\rho_2}\mathcal{O}[z]_{<m_1+m_2} + \ldots + x^{\rho_r}\mathcal{O}[z]_{<m_1+m_2+\ldots+m_r}.$$

Then

$$\underline{E}(\mathcal{F}) = x\mathcal{F}.$$

Proof. (a) We show that $\underline{E}(\mathcal{F}) \subset x\mathcal{F}$. Recall from Lemma 3 that

$$\underline{E}(x^\rho z^i) = x^\rho \cdot [\chi(\rho) z^i + \chi'(\rho) i z^{i-1} + \tfrac{1}{2!}\chi''(\rho) i^{\underline{2}} z^{i-2} + \ldots + \tfrac{1}{n!}\chi^{(n)}(\rho) i^{\underline{n}} z^{i-n}].$$

Therefore, as $\chi^{(j)}(\rho_k) = 0$ for $0 \le j < m_k$, it follows that $\underline{E}$ sends $\mathcal{F}$ into

$$\sum_{k=1}^r x^{\rho_k}\mathcal{O}[z]_{<m_1+\ldots+m_{k-1}} = \sum_{k=2}^r x^{\rho_k}\mathcal{O}[z]_{<m_1+\ldots+m_{k-1}} \subset \sum_{k=2}^r x^{\rho_{k-1}+1}\mathcal{O}[z]_{<m_1+\ldots+m_{k-1}} \subset x\mathcal{F}.$$

Here, we use that $\rho_k - \rho_{k-1} \in \mathbb{N}_{>0}$ and hence $\rho_{k-1} + 1 \le \rho_k$. This proves that $\underline{E}(\mathcal{F}) \subset x\mathcal{F}$.

(b) We show that $\underline{E}(\mathcal{F}) \supset x\mathcal{F}$. It suffices to check that all monomials $x^\sigma z^i \in x\mathcal{F}$ lie in the image, where $\sigma = \rho_k + e$ for some $k = 1, ..., r$ and $e \ge 1$, and where $i < m_1 + \ldots + m_k$. We distinguish two cases.

(i) If $\sigma \notin \Omega$, proceed by induction on i. Let $i = 0$. We have

$$\underline{E}(x^\sigma) = E(x^\sigma) = \chi(\sigma)x^\sigma \neq 0,$$

since σ is not a root of χ. So $x^\sigma \in \underline{E}(\mathcal{F})$. Let now $i > 0$. Lemma 3 yields

$$\underline{E}(x^\sigma z^i) = \chi(\sigma)x^\sigma z^i + \chi^{(j)}(\sigma)x^\sigma \sum_{j=1}^n \frac{i^{\underline{j}}}{j!} z^{i-j}.$$

By the inductive hypothesis and using again that $\chi(\sigma) \neq 0$, we end up with $x^\sigma z^i \in \underline{E}(\mathcal{F})$.

(ii) If $\sigma \in \Omega$, write $\sigma = \rho_k$ for some $1 \le k \le r$. As $x^\sigma z^i = x^{\rho_k} z^i \in x\mathcal{F}$ for $i < m_1 + \ldots + m_k$ and since $\rho_1 < \rho_2 < \cdots < \rho_r$, we know that $k \ge 2$ and

$$x^{\rho_k}z^i \notin x \cdot \sum_{\ell=k}^{r} \mathcal{O}x^{\rho_\ell}[z]_{<m_1+\ldots+m_\ell}.$$

Hence

$$x^{\rho_k}z^i \in x \cdot \sum_{\ell=1}^{k-1} \mathcal{O}x^{\rho_\ell}[z]_{<m_1+\ldots+m_\ell}.$$

This implies in particular that $0 \leq i < m_1 + \ldots + m_{k-1}$, which will be used later on. We proceed by induction on i. Let $i = 0$. By Lemma 3,

$$\underline{E}(x^{\rho_k}z^{m_k}) = \sum_{j=0}^{m_k-1} \frac{(m_k)^{\underline{j}}}{j!}\chi^{(j)}(\rho_k)x^{\rho_k}z^{m_k-j} + \chi^{(m_k)}(\rho_k)x^{\rho_k} = \chi^{(m_k)}(\rho_k)x^{\rho_k}.$$

Here, the sum in the first summand is 0 since ρ_k is a root of χ of multiplicity m_k, and for the same reason, the second summand $\chi^{(m_k)}(\rho_k)x^{\rho_k}$ is non-zero. So $x^\sigma = x^{\rho_k} \in \underline{E}(\mathcal{F})$. Let now $i > 0$ and consider $x^\sigma z^i = x^{\rho_k}z^i \in x\mathcal{F}$. We will use that $i < m_1 + \ldots + m_{k-1}$ as observed above. Namely, this implies that $m_k + i < m_1 + \ldots + m_k$, so that $x^{\rho_k}z^{m_k+i}$ is an element of $\mathcal{F}$. Let us apply $\underline{E}$ to it. Similarly as in the case $i = 0$ we get

$$\underline{E}(x^{\rho_k}z^{m_k+i}) = \frac{(m_k+i)^{\underline{m_k}}}{m_k!}\chi^{(m_k)}(\rho_k)x^{\rho_k}z^i + \sum_{j=m_k+1}^{n} \frac{(m_k+i)^{\underline{j}}}{j!}\chi^{(j)}(\rho_k)x^{\rho_k}z^{m_k+i-j}.$$

The sum appearing in the second summand of the last line belongs to $\underline{E}(\mathcal{F})$ by the induction hypothesis since $m_k + i - j < i$. As $\chi^{(m_k)}(\rho_k) \neq 0$, we end up with $x^\sigma z^i = x^{\rho_k}z^i \in \underline{E}(\mathcal{F})$. This proves that $\underline{E}(\mathcal{F}) = x\mathcal{F}$. ⟲

Example. In the situation of the lemma, the image $\underline{E}(x^{\rho_1}\mathcal{O}[z]_{<m_1})$ will have a gap at $x^{\rho_2}z^i$ for $0 \leq i < m_1$. It will be filled by $\underline{E}(x^{\rho_2}\mathcal{O}[z]_{<m_1+m_2})$ since $\chi^{(m_2)}(x^{\rho_2}) \neq 0$ and consequently $\underline{E}(x^{\rho_2}z^{i+m_2})$ is of the form $c_i x^{\rho_2}z^i$ plus some lower degree terms in z, for some non-zero constant $c_i \in \mathbb{C}$.

Theorem. (Normal form theorem vs3, general case) *Let $L = \sum_{j=0}^n p_j(x)\partial^j \in \mathcal{O}[\partial]$ be an n-th order linear differential operator with holomorphic coefficients p_j in $\mathcal{O}$. Let Ω be a set of local exponents of E with integer differences, ordered increasingly,*

$$\rho_1 < \rho_2 < \ldots < \rho_r,$$

meaning that $\rho_{k+1} - \rho_k \in \mathbb{N}_{>0}$. Let m_k be the multiplicity of ρ_k. Set

$$\mathcal{F} = x^{\rho_1}\mathcal{O}[z]_{<m_1} + x^{\rho_2}\mathcal{O}[z]_{<m_1+m_2} + \ldots + x^{\rho_r}\mathcal{O}[z]_{<m_1+m_2+\ldots+m_r}.$$

Denote by L_0 the initial form of L at 0, and assume that L_0 has shift 0. There exists a linear automorphism $\widehat{u} : \widehat{\mathcal{F}} \to \widehat{\mathcal{F}}$ such that the linear maps $\underline{L}$ and $\underline{L}_0$ on $\widehat{\mathcal{F}}$ induced by L and L_0 satisfy

$$\underline{L} \circ \widehat{u}^{-1} = \underline{L}_0.$$

Moreover, if 0 is a regular singular point of L, then $\widehat{u}$ restricts to a linear automorphism $u : \mathcal{F} \to \mathcal{F}$ such that the linear maps on $\mathcal{F}$ induced by $\underline{L}$ and $\underline{L}_0$ satisfy

$$\underline{L} \circ u^{-1} = \underline{L}_0.$$

Proof. Repeat the proof of version 1 of the normal form theorem, using now Lemma 5 for ensuring the required equality $\underline{L}_0(\mathcal{F}) = x\mathcal{F}$. ⟲

Corollary. *In the situation of the theorem and assuming that L has a regular singularity at 0, let $y_1 = x^\rho, ..., y_{m_\rho} = x^\rho \log(x)^{m_\rho - 1}$, for ρ varying over all local exponents of L, be the basis of solutions of the Euler equation $L_0 y = 0$. Let $\mathcal{F}$ be defined as above with normalizing automorphism $u : \mathcal{F} \to \mathcal{F}$. Then*

$$u^{-1}(y_1) = u^{-1}(x^\rho), ..., y_{m_\rho} = u^{-1}(x^\rho \log(x)^{m_\rho - 1})$$

form a basis of solutions of $Ly = 0$. If $\Omega = \{\rho_1, ..., \rho_r\}$ is an increasingly ordered set of local exponents with integer differences, the solutions $y_{ki} = u^{-1}(x^{\rho_k} \log(x)^i)$, for $1 \le k \le r$, $0 \le i < m_k$, of $Ly = 0$ are of the form

$$y_{\rho_k i}(x) = x^{\rho_k}[f_{ki}(x) + f_{k,i-1}(x)\log(x) + \ldots + f_{k0}(x)\log(x)^i] + \\ + \sum_{\ell=k+1}^{r} x^{\rho_\ell} \sum_{j=m_1+\ldots+m_{\ell-1}}^{m_1+\ldots+m_\ell - 1} h_{kij}(x)\log(x)^j,$$

with holomorphic f_{ki} and h_{kij} in $\mathcal{O}$, where all f_{ki} have non-zero constant term.

Proof. The first part is a direct consequence of the normal form theorem. The explicit description of the solutions in the second part follows by a computation from the formula $u = \mathrm{Id}_\mathcal{F} - S \circ T$ of the normalizing automorphism $u : \mathcal{F} \to \mathcal{F}$. ○

Example. We consider the operator $L = x^2\partial^2 - 2x\partial + 2 + x$ with initial form $L_0 = x^2\partial^2 - 2x\partial + 2$, shift $\tau = 0$, indicial polynomial $\chi(t) = (t-1)(t-2)$ with derivative $\chi'(t) = 2t - 3$, and local exponents $\sigma = 1$, $\rho = 2$ at 0 of multiplicity 1 each. So $\Omega = \{\sigma, \rho\} = \{1, 2\}$. Let $f(x) = \sum_{k=0}^\infty c_k x^k$, $g(x) = \sum_{k=0}^\infty a_k x^k$, $h(x) = \sum_{k=0}^\infty b_k x^k$ with $a_k, b_k, c_k \in \mathbb{C}$ be unknown holomorphic functions. The two prospective (linearly independent) solutions of $Ly = 0$ are of the form

$$\begin{aligned} y_1(x) &= x^2 f(x) = \textstyle\sum_{k=0}^\infty c_k x^{k+2} \in x^2\mathbb{C}\{x\}, \\ y_2(x) &= xg(x) + x^2h(x)\log(x) \\ &= \textstyle\sum_{k=0}^\infty a_k x^{k+1} + \sum_{k=0}^\infty b_k x^{k+2}\log(x) \in x\mathbb{C}\{x\} \oplus x^2\mathbb{C}\{x\}\log(x). \end{aligned}$$

The first one corresponds to the maximal exponent $\rho = 2$, the second to the exponent $\sigma = 1$. It is this second one which interests us. We set $y(x) = xg(x) + x^2h(x)z \in x\mathbb{C}\{x\} \oplus x^2\mathbb{C}\{x\}z$ and consider, according to our preceding constructions, the operator $\underline{L}$ induced by L,

$$\underline{L} : x\mathbb{C}\{x\} \oplus x^2\mathbb{C}\{x\}z \to x\mathbb{C}\{x\} \oplus x^2\mathbb{C}\{x\}z,$$

$$\underline{L}(y(x))) = (L + L'\partial_z)(y(x)) = L(xg(x)) + L'(x^2h(x)) + L(x^2h(x))z.$$

Spliting $\underline{L}$ into two components, according to the direct sum $x\mathbb{C}\{x\} \oplus x^2\mathbb{C}\{x\}z \cong x\mathbb{C}\{x\} \times x^2\mathbb{C}\{x\}$, the map $\underline{L}$ decomposes into $\underline{L} = (L^\sigma + L'^\rho, L^\rho) = (L^1 + L'^2, L^2)$ with linear maps

$$\begin{aligned} &L^1 : x\mathbb{C}\{x\} \to x\mathbb{C}\{x\}, \\ &L'^2 : x^2\mathbb{C}\{x\} \to x\mathbb{C}\{x\}, \\ &L^2 : x^2\mathbb{C}\{x\} \to x^2\mathbb{C}\{x\}. \end{aligned}$$

The analogous decompositions hold for the initial form L_0 of L. We have the formulas

$$L_0^1(x^{1+k}) = \chi(k+1)x^{1+k},$$

$$L_0'^2(x^{2+k}) = \chi'(k+2)x^{2+k},$$

$$L_0^2(x^{2+k}) = \chi(k+2)x^{2+k}.$$

The equation $Ly = 0$ is equivalent to

$$(L_0^1 + L_0'^2 + x)(xg(x), x^2h(x)) = 0$$

and

$$(L_0^2 + x)(xg(x), x^2h(x)) = 0.$$

This just means that

$$\sum_{k=0}^\infty \chi(k+1)a_kx^{k+1} + \sum_{k=0}^\infty \chi'(k+2)b_kx^{k+2} + \sum_{k=0}^\infty a_kx^{k+2} = 0,$$

$$\sum_{k=0}^\infty \chi(k+2)b_kx^{k+2} + \sum_{k=0}^\infty b_kx^{k+3} = 0,$$

say,

$$\sum_{k=0}^\infty (k^2-k)a_kx^{k+1} + \sum_{k=0}^\infty (2k+1)b_kx^{k+2} + \sum_{k=0}^\infty a_kx^{k+2} = 0,$$

$$\sum_{k=0}^\infty (k^2+k)b_kx^{k+2} + \sum_{k=0}^\infty b_kx^{k+3} = 0.$$

Reordering the sums gives

$$\sum_{k=0}^\infty ((k+1)^2-(k+1))a_{k+1}x^{k+2} + \sum_{k=0}^\infty a_kx^{k+2} + \sum_{k=0}^\infty (2k+1)b_kx^{k+2} = 0,$$

$$\sum_{k=0}^\infty ((k+1)^2+k+1)b_{k+1}x^{k+3} + \sum_{k=0}^\infty b_kx^{k+3} = 0,$$

from which we get the following system of linear recurrences ($k \geq 0$)

$$a_k + (k^2+k)a_{k+1} + (2k+1)b_k = 0,$$

$$b_k + (k^2+3k+2)b_{k+1} = 0.$$

We distinguish two cases, $b_0 = 0$ and $b_0 \neq 0$. In the first case, we get $b_k = 0$ for all $k \geq 0$, and from this follows $a_0 = 0$, $a_1 \in \mathbb{C}$ arbitrary, and

$$a_k = -\tfrac{1}{(k-1)^2+k-1}a_{k-1} = -\tfrac{1}{k^2-k}a_{k-1}$$

for $k \geq 2$. We choose $a_1 \neq 0$ in order not to get the trivial zero solution. In the second case, we may take $b_0 \in \mathbb{C}^*$ arbitrary, and then the second set of recurrences implies, for $k \geq 1$,

$$b_k = -\tfrac{1}{(k-1)^2+3(k-1)+2}b_{k-1} = -\tfrac{1}{k^2+k}b_{k-1}.$$

The first set of recurrences then implies $a_0 = -b_0 \neq 0$, $a_1 \in \mathbb{C}$ arbitrary, and, for $k \geq 2$,

$$a_k = -\tfrac{1}{(k-1)^2+k-1}[a_{k-1} + (2(k-1)+1)b_{k-1}] = -\tfrac{1}{k^2-k}[a_{k-1} + (2k-1)b_{k-1}].$$

The first case yields the solution $y_1(x) = x^2g(x)$ with g holomorphic of order 0, corresponding to the maximal exponent $\rho = 2$, the second case the solution $y_2(x) = xg(x) + x^2h(x)\log(x)$ with g and h holomorphic of order 0, corresponding to the smaller exponent $\sigma = 1$.

It may irritate here that two coefficients, namely b_0 and a_1, can be chosen arbitrarily. But in fact, varying a_1 in the expansion of $y_2(x)$ just adds a multiple of the solution $y_1(x)$ to $y_2(x)$: the recursions for a_k are the same in both cases, up to adding $\frac{2k-1}{k^2-k}b_{k-1}$ in the second case.

NOTES PART VII&VIII: ALGEBRAIC SERIES

We will show in this double section that any algebraic power series is *D-finite*, i.e., satisfies a linear differential equation with polynomial coefficients. Further, that any algebraic power series with rational coefficients is *almost integral* (or: *globally bounded*), i.e., has integer coefficients if one multiplies the variable x with a suitable positive integer.

Proposition DIFF. (Abel 1827, Cockle 1860, Harley 1862) *Let $y(x) \in \mathbb{C}[[x]]$ be an algebraic power series. There exists a differential equation $Ly = 0$ with polynomial coeffficients which annihilates $y(x)$ and all its conjugates, that is, $y(x)$ is differentially finite. An equation of minimal order can be constructed from the minimal polynomial P of $y(x)$ or, alternatively, from the set of conjugates of $y(x)$. Its order is the dimension of the $\mathbb{C}$-vectorspace spanned by the conjugates of $y(x)$.* [Stan1, Thm. 2.1, p. 178, Stan3, vol II, Thm. 6.4.6, p. 190, Comt, p. 267, CSTU]

Remarks. (a) The statement holds more generally for algebraic Puiseux series $y(x) = \sum_{k=0}^{\infty} a_k x^{k/e}$, $e \geq 1$, with the same proof, and also over arbitrary fields of characteristic 0. Matzat claims that the minimal differential equation of an algebraic power series in $\mathbb{Q}[[x]]$ has 0 as a non-singular point [Matz, p. 684-685]. Note here that this is not the case for algebraic Puiseux series, as $y(x) = \sqrt{x}$ has differential equation $xy' - \frac{1}{2}y = 0$ with regular singularity at 0.

(b) Already Frobenius shows that if the conjugates of an algebraic series span a $\mathbb{C}$-vectorspace of dimension m then they satisfy a differential equation of order m [Frob2, p. 242].

(c) The construction of L given in the two proofs is not as explicit as one would wish. No direct formula for L in terms of the minimal polynomial P of $y(x)$ seems to be known, not a characterization of the differential equations which can arise in this way.

First proof. If $P(x, y)$ is the minimal polynomial of $y(x)$, it is irreducible in $K(x)[y]$, hence P and $\partial_y P$ have no common factor in $K(x)[y]$. Choose $A, B \in K(x)[y]$ such that $AP + B\partial_y P = 1$. It follows that $B(x, y(x)) \cdot \partial_y P(x, y(x)) = 1$ and hence $\frac{1}{\partial_y P(x,y(x))} \in K(x)[y(x)]$. Differentiating $P(x, y(x)) = 0$ with respect to x yields

$$y'(x) = -\frac{\partial_x P(x, y(x))}{\partial_y P(x, y(x))} \in K(x)[y(x)].$$

By the same argument with $y(x)$ replaced by $y'(x)$ (which is again algebraic over $K(x)$) we get that $y''(x) \in K(x)[y'(x)] \subset K(x)[y(x)]$. Iteration yields $y^{(k)}(x) \in K(x)[y(x)]$. But $K(x)[y(x)]$ is generated over $K(x)$ by $1, y(x), ..., y(x)^m$ with $m = \deg_y P - 1$, and is hence a $K(x)$-vectorspace of dimension less than or equal to the degree of algebraicity of $y(x)$. This implies that $y(x)$ and its the successive derivatives $y'(x), y''(x), ..., y^{(m)}(x)$ are linearly dependent over $K(x)$, so there exists a differential operator $L \in K[x][\partial]$ of order $\leq m$ with $Ly(x) = 0$. $\circlearrowleft$

*Second proof.** Let $S_P(x)$ be a splitting field of P over $\mathbb{C}(x)$. We may choose $S_P(x)$ inside $\mathrm{Puis}((x))$. The differentiation in $\mathrm{Puis}((x))$ with respect to x restricts to a derivation $\widetilde{\partial}$ on $S_P(x)$ since the derivative $y'(x)$ of a root $y(x)$ of P in $S_P(x)$ is a rational function of x and $y(x)$ and thus belongs again to $S_P(x)$: to see this, it suffices to derive $P(x, y(x)) = 0$ with respect to x, $\partial_x P(x, y(x)) + \partial_y P(x, y(x))y'(x)$ and to express the resulting inner derivative $y'(x)$ in terms of x and $y(x)$.

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* We are indebted to Michael Singer for explaining to us this construction of L [e-mail October 30, 2020], see also [CSTU, p. 356].

Alternatively, by the theory of differential fields, one may also use that the derivative on $\mathbb{C}(x)$ extends uniquely to splitting fields of polynomials P in $\mathbb{C}(x)[y]$, see [vdPS, ex. 1.5.3(c), p. 5]: We will show this only for extensions $\mathbb{C}(x)[y]/\langle P(x,y)\rangle$ where P is irreducible. The argument works for any finite extension of a differential field. As $\partial_y P$ is relatively prime to P in $\mathbb{C}(x)[y]$, we may choose $A, B \in \mathbb{C}(x)[y]$ such that $AP + B\partial_y P = 1$. Define via $\partial y = -B(x,y)\partial_x P(x,y)$ a derivation on $\mathbb{C}(x)[y]$ extending ∂_x on $\mathbb{C}(x)$. It sends by construction P to the ideal $\langle P\rangle$ and thus induces a derivation on $\mathbb{C}(x)[y]/\langle P(x,y)\rangle$ as required. This proves the existence. Uniqueness goes as follows: Let $\overline{y}$ denote the residue class of y in $\mathbb{C}(x)[y]/\langle P(x,y)\rangle$. It is algebraic over $\mathbb{C}(x)$ with minimal polynomial P. Let δ be another derivation on $\mathbb{C}(x)[y]/\langle P(x,y)\rangle$ fixing $\mathbb{C}(x)$. Then $\varepsilon = \partial - \delta$ is zero on $\mathbb{C}(x)$. Write $P = \sum_{i=0}^d P_i(x)y^i$. We get

$$\varepsilon(P) = \sum_{i=0}^d \varepsilon(P_i(x))y^i + \sum_{i=0}^d P_i(x)\varepsilon(y^i) = \varepsilon(y)\sum_{i=0}^d iP_i(x)y^{i-1}.$$

From $P(x,\overline{y}) = 0$ and the minimality of P with respect to $\overline{y}$ now follows that $\varepsilon(\overline{y}) = 0$. This proves uniqueness. Iterating this process for the successive extensions from $\mathbb{C}(x)$ to $S_P(x)$, we see that there exists a unique extension of the derivation ∂_x on $\mathbb{C}(x)$ to $S_P(x)$.

We continue with the construction of the minimal differential operator annihilating an algebraic power series. Choose a $\mathbb{C}$-basis $y_1, ..., y_n$ of the vectorspace $V_P(x) \subseteq S_P(x)$ generated by the roots of P (to easen the notation, we suppress the dependence of y_i on x). We have $V_P(x) = \bigoplus_{i=1}^n \mathbb{C}y_i \subseteq S_P(x) = \mathbb{C}(x, y_1, ..., y_n)$. In general, n will be smaller than the degree $\deg P$ of P. The Wronskian matrix of n series $y_1, ..., y_n$ is

$$W(y_1, ..., y_n) = \begin{pmatrix} y_1 & \cdots & y_n \\ y_1' & \cdots & y_n' \\ \vdots & & \vdots \\ y_1^{(n-1)} & \cdots & y_n^{(n-1)} \end{pmatrix}.$$

Denote by $w(y_1, ..., y_n) = \det\, W(y_1, ..., y_n)$ its determinant. Let y be a variable, denote by $y', ..., y^{(n)}$ its formal derivatives and consider the Wronskian

$$W(y, y_1, ..., y_n) = \begin{pmatrix} y & y_1 & \cdots & y_n \\ y' & y_1' & \cdots & y_n' \\ \vdots & \vdots & & \vdots \\ y^{(n-1)} & y_1^{(n-1)} & \cdots & y_n^{(n-1)} \\ y^{(n)} & y_1^{(n)} & \cdots & y_n^{(n)} \end{pmatrix}$$

with determinant $w(y, y_1, ..., y_n)$. Now, for $y_1, ..., y_n$ a basis of V_P, define the n-th order differential operator $L \in \mathbb{C}(x, y_1, ..., y_n)[\partial] = S_P(x)[\partial]$ by

$$Ly = \frac{w(y, y_1, ..., y_n)}{w(y_1, ..., y_n)}.$$

Clearly, L annihilates $y_1, ..., y_n$, by the properties of the Wronskian, and $L \in \mathbb{C}\langle x\rangle[\partial]$. We will show that $L \in \mathbb{C}(x)[\partial]$, i.e., that the coefficients of L are rational functions, and then, after multiplication of L with their common denominator, even polynomials.

The action of the Galois group G of $S_P(x)$ over $\mathbb{C}(x)$ extends trivially to $S_P(x)(y, y', ..., y^{(n)})$, leaving the variable y and its derivatives $y^{(i)}$ fixed. Note that the action of G commutes with differentiation (with respect to x) in $S_P(x)$,

$$(\sigma y_i)' = \sigma y_i'.$$

Indeed, the derivation δ on $S_P(x)$ given by $z \to \sigma^{-1}(\sigma z)'$ equals, when restricted to $\mathbb{C}(x)$, the differentiation ∂ of $\mathbb{C}(x)$. By the uniqueness of the extension $\widetilde{\partial}$ of ∂ to $S_P(x)$ we get $\delta = \widetilde{\partial}$. [•• ... one may also extend the action of G to $\mathrm{Puis}((x))$ and then restrict to $S_P(x)$].

Let $\sigma \in G$ be a group element, and denote by $[\sigma]$ the matrix of $\sigma \in \mathrm{GL}_n(\mathbb{C}(x))$ with respect to the basis $y_1, ..., y_n$ of $V_P(x)$. We claim that $\sigma L = L$. We have

$$\sigma w(y, y_1, ..., y_n) = w(\sigma y, \sigma y_1, ..., \sigma y_n) = w(y, \sigma y_1, ..., \sigma y_n) = \det[\sigma] w(y, y_1, ..., y_n)$$

and

$$\sigma w(y_1, ..., y_n) = w(\sigma y_1, ..., \sigma y_n) = \det[\sigma] w(y_1, ..., y_n),$$

which gives the claim. As this holds for all $\sigma \in G$, it follows that $L \in \mathbb{C}(x)[\partial]$. So we have found a differential equation $Ly = 0$ which is satisfied by all roots of P.

Let us show that it is minimal. Let L' be any other linear differential operator annihilating one of the roots of P. The elements of G act transitively on the roots of P. As they commute with the derivation on $S_P(x)$ we get that L' annihilates all roots of P, hence $V_P(x)$. We divide L' by L inside $\mathbb{C}(x)[\partial]$, $L' = ML + N$ with N an operator of order $< n$, the order of L. But also N annihilates $V_P(x)$, which has $\mathbb{C}$-dimension n. Hence $N = 0$ and L' is a (left)-multiple of L. This concludes the second proof. ◯

Remark. The first proof does not give an explicit formula for the differential equation satisfied by $y(x)$. It shows, however, that the order of L is less than or equal to the degree of algebraicity of $y(x)$. The order of L may be much smaller, as show the examples where the dimension of the vectorspace generated by the conjugates of $y(x)$ is smaller that the degree of algebraicity. In [CSTU], various algorithms are discussed of how to determine the minimal differential operator L annihilating all roots of a polynomial $P \in K(x)[y]$. Note that L need not be irreducible, see [CSTU, Thm. 5.1, p. 376, Prop. 5.4, p. 378, Ex. 5.6, p. 385, Ex. 5.15, p. 391].

Examples. (R5) First order equations $y' - ry = 0$ can be rewritten $\log(y)' = r$ with solution $y(x) = \exp(R(x))$, where R is a primitive of r. If $y(x)$ is a rational function, $r(x) = \frac{y'(x)}{y(x)}$ will be rational as well, with only simple poles. We will investigate later on for which r the solution $y(x)$ is a rational function or an algebraic series. S. Yurkevich showed recently that if $r \in \mathbb{Q}(x)$ and $y' - ry = 0$ has a power series solution $y(x) \in \mathbb{Z}[[x]]$ with integer coefficients, then $y(x)$ is already algebraic. In fact, $y(x)$ will then be the m-th root of a rational function, for some $m \geq 1$.

(R6) Let $P \in \mathbb{C}(x)[y]$ be an irreducible polynomial of degree $d \geq 2$. Assume that the coefficient $a := a_{d-1}(x)$ of y^{d-1} in the expansion of P is non-zero. Set $L = \partial - \frac{a'}{a}$. This is a first order operator with rational coefficients. Let L' be any operator annihilating the roots $y_1, ..., y_d$ of P in $\mathrm{Puis}(x)$. Write $L' = ML + N$ with N an operator of order 0. As L and L' annihilate $z = y_1 + \ldots + y_d$ [•• ... ?] we get $N = 0$, hence $L' = ML$ is a multiple of L.

(R7) Let $y(x) = \sqrt{1+x} + \sqrt{2+x}$. This is an algebraic series with minimal polynomial $P = (y^2 - 2x - 3)^2 - 2(x^2 + 3x + 2)$ of degree 4. Its roots are $\pm\sqrt{1+x} \pm \sqrt{2+x}$, spanning the two-dimensional $\mathbb{C}$-vectorspace $V_P(x) = \mathbb{C}\sqrt{1+x} + \mathbb{C}\sqrt{2+x}$. By what we said earlier, we thus know that the minimal operator annihilating the roots of P has order 2. The coefficient $a_3(x)$ of P is 0, so we are not in the situation of example R6. But both $L_1 = \partial - \frac{1}{1+x}$ and $L_2 = \partial - \frac{1}{2+x}$ divide L from the right, $L = M_1 L_1 = M_2 L_2$. So L is reducible. Actually, it is the least common left multiple of L_1 and L_2.

(R8) The divergent series $y(x) = \sum_{k=0}^{\infty} k!x^{k+1} = x + x^2 + 2x^3 + 6x^4 + 24x^5 + \ldots$ (example 6' in the Introduction) is differentially finite with minimal equation $Ly = x^3y'' + (x^2 - x)y' + y = 0$. The series is integral and transcendent (since divergent), and 0 is an irregular singularity of L. The initial form at 0 is given by the first order operator $L_0 = -x\partial + 1$, while L has order 2. Note that the function $y_1(x) = \exp(-\frac{1}{x})$ is also a solution of $Ly = 0$. Further, $y_2(x) = \exp(-\frac{1}{x}) \cdot Ei(-\frac{1}{x})$ is a second, linearly independent solution, with $Ei(x) = \displaystyle\int_{-\infty}^{x} \frac{e^t}{t} dt = -\gamma - \log(x) - \sum_k (-1)^k \frac{x^k}{k \cdot k!}$ the exponential integral, and γ the Euler-Mascheroni constant. To prove that $y_2(x)$ is a solution, use $Ei'(x) = -\frac{1}{x}\exp(-x)$.

The expansion at infinity is (see example (I11) in the introduction)

$$\widetilde{L} = \partial^2 + [2x + x^4 - x^3]\partial + x^7.$$

Funny enough, $y_1(\frac{1}{x}) = \exp(-x)$ is not a solution of the equation $\widetilde{L}y = 0$, nor $y(\frac{1}{x}) = \sum_{k=0}^{\infty} \frac{k!}{x^{k+1}}$.

(R9) In [Stan1, remark (g), p. 186] Stanley asks for an algorithm which should allow one to decide whether a differentially finite power series $y(x)$ is algebraic. In general, by studying the asymptotics of the series, transcendence is easier to prove. The series $y(x) = \sum_{k=0}^{\infty} \binom{2k}{k}^{2m}$ is transcendental for all integers $m \geq 1$; responding to a question of Stanley in [loc.cit.] it is proven in [ABD, p. 5] that also $z(x) = \sum_{k=0}^{\infty} \binom{2k}{k}^{2m+1}$ is transcendental.

The differential equations for $y(x)$ and $m = 1, 2, 3, 4$ are given by the irreducible operators

$m = 1$: $L = (1 - 4x)\partial - 2$,

$m = 2$: $L = (x - 16x^2)\partial^2 + (1 - 32x)\partial - 4$,

$m = 3$: $L = (x^2 - 64x^3)\partial^3 + (3x - 288x^2)\partial^2 + (1 - 208x)\partial - 8$

$m = 4$: $L = (x^3 - 256x^4)\partial^4 + (6x^2 - 2048x^3)\partial^3 + (7x - 3712x^2)\partial^2 + (1 - 1280x)\partial - 16$.

For $m = 3$, the operator L is the symmetric square of $L_1 = (64x^2 - x)\partial^2 + (96x - 1)\partial + 4$, and the solutions of $Ly = 0$ are hypergeometric series of the form

$y_1(x) = (F(1/4, 1/4; 1/2; 1 - 64x)^2$,

$y_2(x) = F(3/4, 3/4; 3/2; 1 - 64x)^2 \cdot (64x - 1)$,

$y_3(x) = F(1/4, 1/4; 1/2; 1 - 64x) \cdot F(3/4, 3/4; 3/2; 1 - 64x) \cdot \sqrt{64x - 1}$.

(R10) Take the polynomial $P(x, y) = x^2y - x^4 - y^4$ defining an algebraic curve $X \subset \mathbb{C}^2$. The origin is a singular point of X, with tangent cone defined by $x^2y = 0$. It turns out that X has two local analytic branches at 0, one smooth and tangent to the x-axis, the other singular with vertical tangent and cusp singularity isomorphic to $x^2 - y^3 = 0$, see Fig. CTBA.

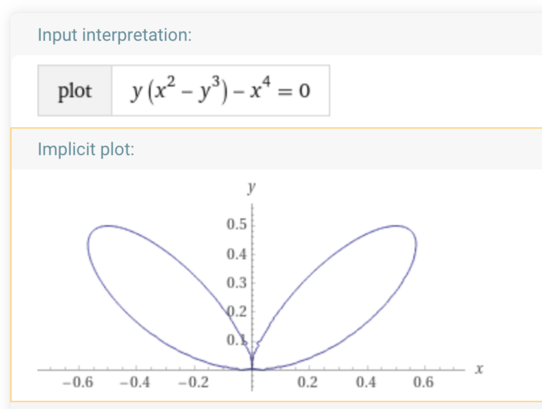

Figure CTBA: The real plane algebraic curve with equation $x^2y = x^4 + y^4$.

The four solutions $y(x)$ of $P(x, y) = 0$ define local parametrizations of the branches. Computations with Maple yield

$$y_1(x) = x^2 + x^6 + 4x^{10} + 22x^{14} + 140x^{18} + O(x^{22}),$$

$$y_{234}(x) = \xi x^{\frac{2}{3}} - \tfrac{1}{3}x^2 - \tfrac{2}{9}\xi^2 x^{\frac{10}{3}} - \tfrac{20}{81}\xi x^{\frac{14}{3}} - \tfrac{1}{3}x^6 + O(x^{\frac{22}{3}}),$$

where ξ is a third root of unity. The minimal differential equation is

$$(27x^3 - 256x^7)y''' - (81x^2 + 768x^6)y'' + 141xy' - 120y = 0.$$

It can be proven that the equation is a symmetric square.

(R11) Take the polynomial $P(x, y) = xy - x^2 - y^4$ defining an algebraic curve $Y \subset \mathbb{C}^2$. The origin is a singular point of X, with tangent cone defined by $x(y - x) = 0$. It turns out that X has two local analytic branches at 0, both smooth and tangent to the y-axis, respectively, the first diagonal, see Fig. CTBB.

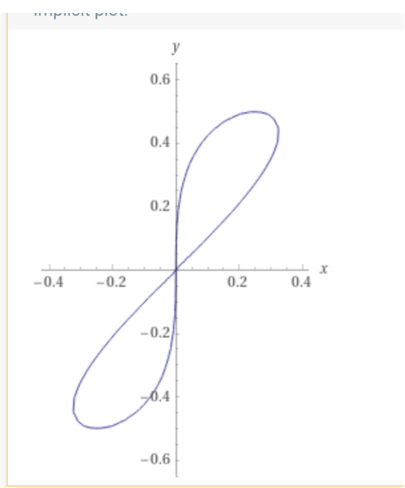

Figure CTBB: The real plane algebraic curve with equation $xy = x^2 + y^4$.

The roots are the (algebraic) series

$$y_1(x) = x + x^3 + 4x^5 + 22x^7 + 140x^9 + 969x^{11} + 7084x^{13} + 53820x^{15} + 420732x^{17} + O(x^{19})$$

and

$$y_2(x) = x^{\frac{1}{3}} - \tfrac{1}{3}x - \tfrac{2}{9}x^{\frac{5}{3}} - \tfrac{20}{81}x^{\frac{7}{3}} - \tfrac{1}{3}x^3 - \tfrac{364}{729}x^{\frac{11}{3}} - \tfrac{5236}{6561}x^{\frac{13}{3}} - \tfrac{4}{3}x^5 - \tfrac{135850}{59049}x^{\frac{17}{3}} + O(x^{\frac{19}{3}}).$$

The minimal differential equation is

$$(27x^3 - 256x^5)y''' - 768x^4y'' + (15x - 192x^3)y' - 15y = 0.$$

Its unique power series solution is

$$y(x) = x \cdot F(\tfrac{-1}{24}, \tfrac{5}{24}; \tfrac{2}{3}; \tfrac{256}{27}x^2) \cdot F(\tfrac{7}{24}, \tfrac{13}{24}; \tfrac{4}{3}; \tfrac{256}{27}x^2),$$

the other two solutions being Puiseux series. The equation is a symmetric square, for instance of

$$(108x^2 - 1024x^4)y'' - 1024x^3y' + (15 + 64x^2)y = 0.$$

(R12) In the example of [Stan1, ex. 2.5, p. 179] it is mentioned that the secans function $y(x) = \sec(x) = \frac{1}{\cos(x)}$ is not differentially finite, whereas its inverse inverse $z(x) = \cos(x)$ is differentially finite with equation $z'' + z = 0$.

Remarks. (a) It follows from Prop. DIFF that the minimal differential equation of an algebraic power series has only algebraic power series solutions: a transcendent power series solution or logarithms cannot appear. However, the minimal differential equation of a transcendent differentially finite series may also have algebraic solutions, see example 9 of Ke-Ying and Jinzhi from section P. The defining operator is then necessarily reducible, in view of Prop. IRR.

(b) [Singer, mail oct 30 2020] The second proof of Prop. DIFF shows that the minimal differential operator L annihilating an algebraic power series $y(x)$ is reducible if and only if the $\mathbb{C}$-vectorspace $V_P(x)$ spanned by the conjugates of $y(x)$ contains a proper non-zero subspace W which is invariant under the action of the Galois group G of the splitting field $S_P(x)$ of the minimal polynomial P of $y(x)$. [•• ... Actually, this algebraic Galois group G coincides with the differential Galois group associated to the Picard-Vessiot extension of the annihilating operator L. Here, the Picard-Vessiot extension is defined as a minimal extension of $\mathbb{C}(x)$ containing all solutions of the differential equation $Ly = 0$. It coincides in the present setting with the splitting field of the polynomial P.]

(c) [•• ... In Prop. ONE we have seen for order one equations $y' = ry$ with r rational function how to deduce the existence of an algebraic solution if one knows that a power series solution with integer coefficients exists. One may want to apply a similar method to order two equations, $y'' = ry' + sy$ with rational functions r and s. A special and possibly easier case would be equations of the form $y'' = sy$, without first order derivative.

Let us check whether a second order Euler equation $x^2y'' + cxy' + dy = 0$ can be brought into the form $x^2y'' + dy = 0$. This latter equation has indicial polynomial $\chi = r(r-1) + d = r^2 - r + d$. Set $y = x^a z$ and substitute for y in the equation. We get

$$x^2[a(a-1)x^{a-2}z + ax^{a-1}z' + x^a z''] + cx[ax^{a-1}z + x^a z'] + dx^a z = 0,$$

say,

$$a(a-1)x^a z + ax^{a+1}z' + x^{a+2}z'' + c[ax^a z + x^{a+1}z'] + dx^a z = 0,$$

$$x^{a+2}z'' + (a+c)x^{a+1}z' + (ca + a(a-1) + d)x^a z = 0.$$

Setting $a = -c$, say, $z = x^c y$, gives

$$x^{2-c}z'' + (-c^2 + c(c+1) + d)x^{-c}z = 0,$$

and then, multiplying with x^c,

$$x^2 z'' + (c+d)z = 0.$$

An algebraic power series $h(x)$ with $h(0) = 0$ is called *étale algebraic* if its minimal polynomial $P(x, y)$ satisfies the assumption $\partial_y P(0, 0) \neq 0$ of the implicit funtion theorem. It is then the *unique* power series solution of $P(x, y) = 0$ at 0. It can be constructed iteratively up to any degree by Newton's algorithm. The respective definition also applies for algebraic series with non-zero constant term, assuming that $\partial_y P(0, h(0)) \neq 0$. The next lemma says that any algebraic series becomes étale algebraic after clipping off its expansion up to sufficiently high degree. It also holds for series in several variables.

Simple Root Lemma. (one variable case) *For any algebraic series $h(x) = \sum_{k=0}^{\infty} c_k x^k$ in one variable over a field of characteristic 0, there is an integer $e \geq 1$ such that the series $a(x) = \sum_{k=e+1}^{\infty} c_k x^{k-e}$ is a simple root of its minimal polynomial. Said differently, every univariate algebraic series decomposes into a sum*

$$h(x) = k(x) + x^e \cdot a(x)$$

of a polynomial k and a monomial multiple $x^e \cdot a$ of an étale algebraic series a.

Remarks. (a) For the proof it is convenient to choose the decomposition such that $a(0) = 0$.

(b) For e one can choose the order of $\partial_y P(x, h(x))$. How does it relate to the degree of the minimal polynomial without knowing h? Is there an effective way to determine e?

Proof. Let $P(x, y)$ be the minimal polynomial of h. It has minimal degree, hence its partial derivative $\partial_y P$ has non-zero evaluation at h, say, $\partial_y P(x, h(x)) \neq 0$ [... in positive characteristic $p > 0$, it could happen that $\partial_y P$ is identically zero, but then P would have been a polynomial in y^p as e.g. $y^p - (1 + x)$.]. Let $e \geq 0$ be the order of $\partial_y P(x, h(x))$. Write $h(x) = k(x) + a(x) \cdot x^e$ with a polynomial $k(x)$ of degree $\leq e$ (the truncation of h at degree e) and a series $a(x)$ vanishing at 0, $a(0) = 0$. Taylor expansion gives

$$0 = P(x, h) = P(x, k + a \cdot x^e) = P(x, k) + \partial_y P(x, k) \cdot a \cdot x^e + S(x, a \cdot x^e),$$

with $S(x, y)$ a polynomial of order at least two in y. Now observe that

$$\partial_y P(x, k) = \partial_y P(x, h - a \cdot x^e) = \partial_y P(x, h) - \partial_y^2 P(x, h) \cdot a \cdot x^e + T(x, a \cdot x^e),$$

with $T(x, y)$ a polynomial of order at least two in y. As $a(0) = 0$ and hence $\mathrm{ord}(a \cdot x^e) > e$, and as $\mathrm{ord}(T(x, a \cdot x^e)) > e$, it follows that $\partial_y P(x, k)$ and $\partial_y P(x, h)$ have the same order e. From the first displayed equation we now conclude that $P(x, k)$ has order at least $2e$. Write it as $P(x, k) = x^{2e} \cdot R(x)$ for some polynomial $R(x)$. Divide the first displayed equation by x^{2e} and get

$$a + Q(x, a) + R(x) = 0$$

with $Q(x, y) = x^{-2e} \cdot S(x, y \cdot x^e)$ a polynomial of order at least two in y. Thus the series $a(x) = x^{-e} \cdot (h(x) - k(x))$ is a simple root of the equation

$$\widetilde{P}(x, y) := y + Q(x, y) + R(x) = 0.$$

Remarks. (a) The attentive reader will observe that the proof is a predecessor of the proof of the Artin approximation theorem given below.

(b) If the series h is defined over $\mathbb{Q}$, the polynomials $\widetilde{P}$ and $\widetilde{Q}$ will involve finitely many rational coefficients. Replacing then x by an integer multiple $L \cdot x$ with L having sufficiently many prime divisors (e.g., taking for L the least common multiple of all occurring denominators of $\widetilde{P}$ and $\widetilde{Q}$), one achieves via $\widetilde{P}(L^2 \cdot x, Ly)$

a minimal polynomial for $a(L \cdot x)$ with integer coefficients. This observation will be useful in the proof of the Eisenstein theorem below.
(c) It seems that the above proof does not use that K has characteristic zero.
(d) Algebraic series with complex coefficients are holomorphic.

Eisenstein's theorem (Eisenstein 1852) [Hei, Herm, Sus] *Let $h(x) = \sum c_k x^k \in \mathbb{Q}\langle x\rangle$ be an algebraic series in one variable with rational coefficients $c_k \in \mathbb{Q}$.*
(i) *The denominators of the coefficients a_k have only finitely many prime divisors.*
(ii) *There exists a non-zero integer $\ell \in \mathbb{N}$ so that $h(\ell \cdot x) \in \mathbb{Z}\langle x\rangle$ has integer coefficients, i.e., h is globally bounded.*

Remarks. (a) Clearly, assertion (ii) implies (i). Eisenstein stated the theorem without proof in 1852 in the case where h is an étale algebraic series. Heine proposed in 1853 a proof which seems to be incomplete, and then gave in [...] a rigorous proof, cited by Pólya and Szegö in [PoSz]. They also cite Weierstrass. See also [Ber] for an alternative proof within classical algebraic geometry and using the Riemann-Roch theorem. Other proofs were proposed by Hermite and Susák.
(b) We have already seen that rational series are globally bounded. The example $\sqrt{1+x}$ of the introduction is also globally bounded.
(c) There are various attempts to bound the smallest integer ℓ, the so called Eisenstein bound [DvP, Schm, BiBo, DwR].
(d) The theorem implies that univariate algebraic series with rational coefficients which are not polynomials have finite radius of convergence. Indeed, a series in $\mathbb{Z}\langle x\rangle$ is either a polynomial or has radius of convergence ≤ 1.
(e) Denef and Lipshitz prove in [DeLi1, Thm. 6.2, page 60] that any algebraic series in n variables is the diagonal of a rational series in $2n$ variables. From this follows immediately Eisenstein's theorem in several variables, see below. Safonov [Saf1] states and seems to prove the multivariate version of Eisenstein's theorem using the description of multivariate algebraic series as certain diagonals of rational series in just one more variable.

Proof. We give two proofs. (i) (via implicit function theorem) By the univariate Simple Root Lemma we may assume (after truncation at a suitable degree and division of the remainder series by a monomial) that h is an étale algebraic series. Write the minimal polynomial of h as $P(x, y) = \ell y + Q(x, y) + R(x)$, for $\ell \in \mathbb{Z} \setminus \{0\}$, with polynomials Q and R with integer coefficients and such that Q is at least quadratic with respect to y.
After substitution of x by $\ell^2 \cdot x$ and multiplication of y by ℓ we may assume that P is in fact of the form $P(x, y) = y + Q(x, y) + R(x)$ with polynomials Q and R with integer coefficients. The Newton algorithm described to construct the solution of $P(x, y) = 0$ at 0 then yields the series $h(x)$. It must have integer coefficients since no divisions occur. This is what had to be shown.

(ii) (via diagonals) One may also use Furstenberg's theorem: Every univariate algebraic series $h(x)$ with coefficients in $\mathbb{Q}$ is the diagonal of a rational series in two variables: $r(y, z) = \sum_{ij} c_{ij} y^i z^j$, $\mathrm{diag}(r)(x) := \sum_i c_{ii} x^i$. It is easy to see that rational series are globally bounded, and hence also their diagonals are. $\circlearrowleft$