EXERCISES FUCHSIAN DIFFERENTIAL EQUATIONS FALL 2022

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33 Let $r \in \mathbb{C}(x)$ or $\mathbb{Q}(x)$ be a rational function in one variable x. Describe a primitive

$$s = \int r \text{ of } r$$

by using the Hermite decomposition of r, say, the partial fraction expansion.

Ref. https://en.wikipedia.org/wiki/Partial_fraction_decomposition

34 (a) Let a and b be power series in $\mathbb{Z}[[x]]$ such that

$$e^{a(x)} \cdot b(x) \in \mathbb{Z}[[x]]$$

is a power series with integer coefficients. What can you conclude about a and b?

(b) Solve $y' = y$ in characteristic $p = 2$ in the differential ring $\mathbb{F}_2(z_1, z_2, ...)((x))$, where

$$(z_{k+1})' := \frac{1}{x} \cdot \frac{1}{z_1 \cdots z_k}.$$

(c) Show that, in characteristic 0, the iterated logarithm $\log^{[k]}(x) := \log(\cdots(\log(x))\cdots)$ satisfies the same differentiation rule as in (b).

35 Let $r \in \mathbb{Q}(x)$ be a rational function. Try to solve explicitly

$$y'' = ry.$$

Remark. Hopefully you can find at least some interesting specific r's for which you can solve the equation. Then check for these equations whether the solutions found have integer coefficients.

36 Christmas Challenge: Prove Bézivin's conjecture for order 1 differential equations.

If $y(x) \in \mathbb{Z}[[x]]$ is a solution with integer coefficients of a first order equation

$$y' = r(x)y,$$

for $r \in \mathbb{Q}(x)$ a rational function, then $y(x)$ is already an algebraic series.