EXERCISES FUCHSIAN DIFFERENTIAL EQUATIONS FALL 2022

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5 Determine the initial forms and the local exponents of the Gaussian hypergeometric differential equation

$$x(x-1)y'' + [(a+b+1)x - c]y' + aby = 0$$

at all its singular points, $a, b, c \in \mathbb{Q}$, $c \notin \mathbb{Z}_{\leq 0}$. Don't forget to check at $x = \infty$. [van der Waall, p. 12].

6 Rewrite a second or third order differential equation $Ly = 0$ as a system $Y' = AY$ of first order equations. Try to express the local exponents of L at 0 in terms of the matrix A. Illustrate your findings in an example.

7 Let $r(x) \in \mathbb{C}(x)$ be a rational function and let $m \in \mathbb{N}$. Then $y(x) = \sqrt[m]{r(x)}$ is an algebraic function (i.e., algebraic over $\mathbb{C}(x)$).

(a) Compute the first few coefficients of the expansion of $y(x)$ for $r(x) = 1 + x$ and $m = 3$.

(b) Find a linear differential equation $Ly = 0$ with polynomial coefficients with solution $y(x)$.

(c) Now forget about $y(x)$ and solve $Ly = 0$ "blindly".

(d) When is 0 a non-singular point of L?

(e) In case that it is singular, do you expect it to be a "regular" singularity?

8 (a) Show that the first few coefficients of the hypergeometric series

$$y(x) = \sum_{k=0}^{\infty} \frac{(30k)!k!}{(15k)!(10k)!(6k)!} x^k$$

are integers.

(b)* All coefficients are integers. (The asterisque * signifies that this is more challenging).

(c)* Try to find a differential equation for $y(x)$.

Remark. This is a very famous example, studied already by Chebychev. See the paper of F. Rodriguez-Villegas "Integral ratios of factorials ..." for a detailed discussion.