

EXERCISES FUCHSIAN DIFFERENTIAL EQUATIONS FALL 2022

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Here are a couple of examples and problems to play with. This will help you to get a better grasp on the material we discuss in our class. Most exercises are very easy, so don't be frightened. If there is interest, we could have each week a separate discussion session for the exercises.

1 Solve the three equations

$$x^3 y''' - 3x^2 y'' + 6xy' - 6y = 0,$$

$$x^3 y''' - x^2 y'' + 2xy' - 2y = 0,$$

$$x^3 y''' + xy' - y = 0.$$

Note. The solutions will be simple functions. Always try to construct a whole basis, not just a single solution. If you get stuck, watch again the trailer of the course for a hint.

2 Find linear differential equations for

$$\exp\left(\frac{1}{x}\right),$$

$$\sqrt[3]{1+x^2},$$

$$x^2 \log(x)^3.$$

Then try to find the other solutions of the obtained equations.

3 (a) Consider the linear recursion

$$c_k = \frac{k}{k+1} c_{k-1} - c_{k-2},$$

with $c_0 = 1$ and $c_1 = 2$. Let $y = \sum_k c_k x^k$ be the associated generating series. Find a linear differential equation for y .

(b) One can prove that $\log(\log(x))$ does not satisfy a linear differential equation (that's not so easy, but maybe you have an idea). Find a non-linear equation (that's again easy).

4 Let $Ly = 0$ be a second order linear differential equation with coefficients in $\mathbb{C}[x]$, say,

$$L = a_2(x)\partial^2 + a_1(x)\partial + a_0(x).$$

Now replace x by $\frac{1}{x}$. How does the equation change? Then show that this allows you to study the solutions of $Ly = 0$ at $x = \infty$ by the solutions of the transformed equation $\tilde{L}y = 0$ at $x = 0$.

Note. As x is replaced by $\frac{1}{x}$ also $\partial = \frac{d}{dx}$ and $\partial^2 = \frac{d^2}{dx^2}$ will change. How?